On Reverse Super Vertex-Magic Labeling

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Abstract—For a graph $G(V, E)$ an injective mapping $f$ from $V \cup E$ to the set $\{1, 2, 3, \ldots, v+\varepsilon\}$ is a reverse vertex-magic labeling if there is a constant $h$ so that for every vertex $v \in V$, $f(v) + \sum f(uv) = h$ where the difference runs over all vertices $u$ adjacent to $v$. A vertex-magic labeling $f$ is called super vertex-magic labeling if $f(E) = \{1, 2, 3, \ldots, \varepsilon\}$ and $f(V) = \{\varepsilon + 1, \varepsilon + 2, \ldots, \varepsilon + v\}$. A graph $G$ is called a reverse super vertex-magic if there exists a reverse super vertex-magic labeling of $G$. In this paper, we established some properties of reverse super vertex magic trees and exhibit reverse super vertex-magic labeling of a kite graph.

Keywords—reverse Vertex-magic labeling, reverse super vertex-magic labeling, kite graph.

I. INTRODUCTION

In this paper, we consider only undirected simple finite graph. The graph $G$ has vertex set $V = V(G)$ and edge set $E = E(G)$ and we take $V = |V(G)|$ and $E = |E(G)|$.

MacDougall, Miller, Slamin and Wallis [1] introduced the notion of a vertex magic total labeling in 1999. For a graph $G(V, E)$ an injective mapping $f$ from $V \cup E$ to the set $\{1, 2, 3, \ldots, v+\varepsilon\}$ is a vertex-magic total labeling if there is a constant $h$ so that for every vertex $v \in V$, $f(v) + \sum f(uv) = k$ where the sum runs over all vertices $u$ adjacent to $v$. A vertex-magic labeling $f$ is called super vertex-magic [2] labeling if $f(E) = \{1, 2, 3, \ldots, \varepsilon\}$ and $f(V) = \{\varepsilon + 1, \varepsilon + 2, \ldots, \varepsilon + v\}$. A graph $G$ is called a super vertex-magic if there exists a super vertex-magic labeling of $G$.

In [5], S.VenkataRamanaetalintroduced the concept of reverse super vertex-magic labeling of a graph. A reverse vertex-magic labeling $f$ is a bijection $f$ from $V \cup E$ onto the integers $\{1, 2, 3, \ldots, v+\varepsilon\}$ such that for all vertex $u$, $f(N(u)) - f(u)$is a constant.

A reverse vertex-magic labeling $f$ is called reverse super vertex-magic labeling if $f(E) = \{1, 2, 3, \ldots, \varepsilon\}$ and $f(V) = \{\varepsilon + 1, \varepsilon + 2, \ldots, \varepsilon + v\}$. A graph $G$ is called reverse super vertex-magic if there exists a reverse super vertex-magic labeling of G.

II. MAIN RESULTS

Theorem 1. No reverse super vertex-magic graph has two or more isolated vertices or an isolated edge.

Proof. If $f$ is a reverse super vertex-magic labeling of a graph $G$ with constant $k$ then any isolated vertex $x$ has a label $f(x) = k$. So, there cannot be two such vertices.

Suppose there is an isolated edge $xy$. Then $f(xy) - f(x) = f(xy) - f(y) = k$. Hence $f(x) = f(y)$ which is a contradiction. Hence there is no isolated edge.

Theorem 2. Let T be a tree with $n$ internal vertices and $tn$ leaves. Then T does not admit a reverse super vertex-magic labeling if $t > \frac{3n+1}{n}$.

Proof:- If T has $n$ internal vertices and $tn$ leaves then $\nu = (t + 1)n$ and $\varepsilon = mn - n - 1$. So the labels used for the edges are $\{1, 2, 3, \ldots, mn - n - 1\}$ and for the vertices are $\{mn, mn + 1, \ldots, 2tn + 2n - 1\}$. The maximum possible sum of weights on the leaves is
\[
[(tn + 2n - 1 + 1) + (tn + 2n - 1 + 2) + \ldots + (tn + 2n - 1 + tn)]
- [(n - 1 + 1) + (n - 1 + 2) + \ldots + (n - 1 + tm)]
= tn(tn + 2n - 1) + \frac{tn(m + 1)}{2} - (n - 1)m + \frac{tn(m + 1)}{2}
= tn(tn + n)
\]
Since there are $tn$ leaves, we get
$tnk \le tn(tn + n)$
$k \le tn + n \rightarrow (1)$

On the other hand, the minimum possible sum of weights on the internal vertices occurs when the smallest labels $\{1, 2, 3, \ldots, n - 1\}$ are assigned to internal edges (because they will be added twice), the remaining edges are assigned to the labels $\{n, n + 1, n + 2, \ldots, \varepsilon\}$ and the remaining vertices are assigned to the labels $\{\varepsilon + 1, \varepsilon + 2, \ldots, \varepsilon + n\}$. Hence the minimum possible sum of weights on the internals is
\[
2(1 + 2 + \ldots + n - 1) + (n + n + 1 + \ldots + \varepsilon)
- [\varepsilon + 1 + \varepsilon + 2 + \ldots + \varepsilon + n]
\]

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\[ \frac{n(n - 1)}{2} + \frac{(\varepsilon + n)(\varepsilon + n + 1)}{2} \\
- 2\left[ n\varepsilon + \frac{n(n + 1)}{2} \right] \\
= \frac{n(n - 1)}{2} + \frac{(m + 2n - 1)(m + 2n)}{2} \\
- 2\left[ n(mn - n + 1) + \frac{n(n + 1)}{2} \right] \\
= \frac{n}{2}[t^2n + (4n - t) + 5n - 3] - [2n(mn - n + 1) + n(n + 1)] \\
= \frac{n}{2}[t^2n + (4n - t) + 5n - 3 - 4tn - 4n + 4 - 2n - 2] \\
= \frac{n}{2}[t^2n - t - n - 1] \\
\]

Since there are \( n \) internal vertices,

\[
k \geq \frac{n}{2}[t^2n - t - n - 1] \\
k \geq \frac{1}{2}[t^2n - t - n - 1] \\
\rightarrow (2)
\]

Therefore no labeling will be possible when

\[
\frac{1}{2}[t^2n - t - n - 1] > m + n
\]

That is, when \( t^2n - (2n + 1)t - (3n + 1) > 0 \)

\[
t > \frac{(2n + 1) + \sqrt{(2n + 1)^2 + 4n(3n + 1)}}{2n} \\
= \frac{2n + 1 + 4n + 1}{2} = \frac{3n + 1}{n}
\]

**Theorem 3.** If \( \varepsilon \) is the largest degree of any vertex in a tree \( T \) with \( " \) vertices and \( " \) edges then \( T \) does not admit a super vertex-magic labeling wherever \( \Delta > \frac{-1 + \sqrt{1 + 16\nu}}{2} \).

**Proof.** Let \( c \) be the vertex of maximum degree \( \varepsilon \). The minimum possible weight of \( c \) is \( \varepsilon + 1 - (1 + 2 + 3 + \ldots + \Delta) \) . Therefore,

\[
k \geq \frac{\Delta(\Delta + 1)}{2} - (\varepsilon + 1) \\
k \geq \frac{\Delta(\Delta + 1)}{2} - \nu \\
\rightarrow (3)
\]

Since there is an internal vertex of degree \( \Delta \) there are at least \( \varepsilon \) leaves in \( T \). So the maximum possible sum of weights on the leaves is at most the sum of the \( \Delta \) largest labels from \( f(E) \) and the \( \Delta \) largest labels from \( f(V) \). Hence,

\[
\Delta k \leq [(\varepsilon + \nu - \Delta + 1) + (\varepsilon + \nu - \Delta + 2) + \ldots + (\varepsilon + \nu - \Delta + \Delta)] \\
- [(\varepsilon - \Delta + 1) + (\varepsilon - \Delta + 2) + \ldots + (\varepsilon - \Delta + \Delta)] \\
= [(\varepsilon + \nu - \Delta) + \frac{\Delta(\Delta + 1)}{2}] - \left[\frac{\Delta(\varepsilon - \Delta) + \Delta(\Delta + 1)}{2}\right] \\
= \Delta v \\
k \leq \nu
\]

So labeling will be impossible whenever \( \nu < \frac{\Delta(\Delta + 1)}{2} - \nu \)

That is, when \( \Delta^2 + \Delta - 4\nu > 0 \)

\[
\Delta > \frac{-1 + \sqrt{1 + 16\nu}}{2} \Rightarrow \Delta \in \mathbb{N}
\]

**Remark.** The following table shows the maximum degree permitted by the restriction given in Theorem 3 for some small values of \( \nu \).

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta )</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Theorems 2 and 3 do not provide sufficient condition for a graph to be a reverse super vertex-magic, since we can prove that there is a tree with 7 vertices and \( \Delta = 3 \) shown in Figure 1, which does not admit any reverse super vertex magic labeling.

![Figure 1](http://www.ijritcc.org)

The reason is as follows: The vertex sum varies from 3 to 9. Since the minimum vertex sum itself is 3, the labels 1 and 2 can be assigned only to the internal edges. Therefore, the vertex sum of \( d \) is 3. The remaining labels 3, 4, 5, 6 are assigned to the edges \( ac, be, ge, gf \). Hence one of the leaves must have a vertex sum 3, which contradicts the fact that vertex sums are consecutive integers.

**Theorem 4.** Let \( G \) be a graph obtained by joining a pendant vertex with a vertex of degree 2 of a comb graph. Then \( G \) admits reverse super vertex-magic labeling.

**Proof.** Let \( V \) be the vertex set \( \{a_1, a_2, a_3, \ldots, a_t\} \cup \{a_{i1}, a_{i2}, a_{i3}, \ldots, a_{it}\} \) and \( E = \{a_1a_2, a_2a_3, \ldots, a_{it}, a_{i1}, a_{i2}, a_{i3}, \ldots, a_{it} : 1 \leq i \leq t - 1\} \).

Here \( \nu = 2t + 1 \) and \( \varepsilon = 2t \). Define \( f : E \rightarrow \{1, 2, 3, \ldots, \varepsilon\} \) as follows

\[
f(a_{i1}) = t - i \quad \text{if} \quad 1 \leq i \leq t - 1
\]
\[ f(a_{i_1}) = t, \]
\[ f(a_{i_2}) = t + i \text{ if } 1 \leq i \leq t \]

The vertex labelings are as follows:
\[ f(a_{i_1}) = 2t + 1 + i \text{ if } 1 \leq i \leq t \]
\[ f(a_{i_2}) = 2t + 1 \]
\[ f(a_{i_3}) = 4t + 2 - i \text{ if } 1 \leq i \leq t \]

It can be easily verified that \( f \) is a reverse super vertex-magic labeling with a reverse vertex-magic constant
\[ k = t + 1. \]

**Example.** Example of a reverse super vertex-magic labeling of a graph \( G \) with \( h = 26 \) is given in Figure 2.

**Figure 2**

\[ \begin{align*}
 &\text{11} \\
 &\text{5} \\
 &\text{21} \\
 &\text{4} \\
 &\text{20} \\
 &\text{3} \\
 &\text{19} \\
 &\text{2} \\
 &\text{18} \\
 &\text{1} \\
 &\text{17} \\
 &\text{12} \\
 &\text{6} \\
 &\text{7} \\
 &\text{8} \\
 &\text{9} \\
 &\text{10} \\
 &\text{13} \\
 &\text{14} \\
 &\text{15} \\
 &\text{16}
\end{align*} \]

**Definition.** An \((n, t)\)-kite graph consists of a cycle of length \( n \) with a \( t \)-edge path (the tail) attached to one vertex of a cycle.

**Theorem 5.** An \((n, t)\)-kite graph admits a reverse super vertex-magic labeling iff \( n + t \) is odd.

**Proof.** Let \( G \) be an \((n, t)\)-kite graph. Let the vertex set \( V = \{v_1, v_2, v_3, \ldots, v_n\} \) and the edge set \( E = \{e_i = v_i v_{i+1}, e_n = v_n v_1 : 1 \leq i \leq n - 1\} \cup \{u_t, u_{t+1}, \ldots, u_n\} \).

Hence \( V = n + t \).

Suppose \( G \) admits a reverse super vertex-magic labeling \( f \) with a reverse super vertex-magic constant \( k \). Then \( k = \frac{n + t - 1}{2} \), as \( k \) is an integer \( V = n + t \) must be odd.

Conversely assume that \( V \) is odd. Hence either \( n \) or \( t \) is odd.

We consider two cases.

**Case (i) \( n \) is odd and \( t \) is even.** Define \( f: V \cup E \to \{1, 2, 3, \ldots, 2n + 2t\} \) as follows: For \( 1 \leq i \leq n \),
\[ f(e_i) = \begin{cases} 
\frac{t + i + 1}{2} & \text{if } i \text{ is odd} \\
\frac{t + n + 1 + i}{2} & \text{if } i \text{ is even}
\end{cases} \]

For \( 1 \leq i \leq t \),
\[ f(e_i) = \begin{cases} 
\frac{t + n + 1 + i + 1}{2} & \text{if } i \text{ is odd} \\
\frac{i}{2} & \text{if } i \text{ is even}
\end{cases} \]

The vertex labelings are as follows:
\[ f(v_i) = n + 2t + i \text{ if } 1 \leq i \leq n, \]
\[ f(u_i) = n + 2t - i \text{ if } 1 \leq i \leq t. \]

It can be easily verified that \( f \) is a reverse super vertex-magic labeling of \( G \) with \( k = \frac{n + t - 1}{2} \).

**Case (ii) \( n \) is even and \( t \) is odd.** We consider two sub cases.

**Subcase (i) \( t > n \).** For \( 1 \leq i \leq n \)
\[ f(e_i) = \begin{cases} 
n - \frac{t - 1}{2} & \text{if } i \text{ is odd} \\
\frac{3n + t + 1 - i}{2} & \text{if } i \text{ is even}
\end{cases} \]

and for \( 1 \leq i \leq t \),
\[ f(e_i) = \begin{cases} 
\frac{i - (t - n)}{2} & \text{if } i = t - n + 2, t - n + 4, \ldots, t \\
\frac{n + i}{2} & \text{if } i \text{ is even}
\end{cases} \]

The vertex labelings are as follows:
\[ f(v_i) = 2t + n + i, \quad 2 \leq i \leq t - n + 1 \]
\[ f(u_i) = \begin{cases} 
2n - 1 + i & 2 \leq i \leq t - n + 2 \\
2n - 1 + i & t - n + 2 \leq i \leq t \]

It can be easily verified that \( f \) is a reverse super vertex-magic labeling of \( G \) with \( k = \frac{n + t - 1}{2} \).

**Subcase (ii) \( t \leq n \).** For \( 1 \leq i \leq n \)
\[ f(e_i) = \begin{cases} 
\frac{t - i}{2} & \text{if } i = 1, 3, \ldots, t - 2 \\
\frac{3t + 2n + 1 - i}{2} & \text{if } i = t, t + 2, \ldots, n \\
\frac{t + n - i}{2} & \text{if } i = 2, 4, \ldots, n
\end{cases} \]
\[ f(x_i) = \begin{cases} \frac{t - i}{2} & \text{if } i = 2, 4, \ldots, t - 1, 1 \leq i \leq t \vspace{10pt} \\
\frac{3t + n + 1}{2} - \frac{i + 1}{2} & \text{if } i = 1, 3, \ldots, t, 1 \leq i \leq t \end{cases} \]

The vertex labelings are as follows:
\[ f(v_i) = 2n + t + 1. \]
\[ f(v_i) = \begin{cases} n + 2t - i & \text{if } 2 \leq i \leq t - 1 \\
3n + 2t - i & \text{if } t \leq i \leq n \end{cases} \]
\[ f(u_i) = n + t + 4 \]
\[ f(u_i) = 2n + t + 2 - i, \text{ if } 2 \leq i \leq t. \]

It can be easily verified that \( f \) is a super vertex-magic labeling with \( k = \frac{n + t - 1}{2} \).

**Example.** Example of a super vertex-magic labeling of a kite graph with \( n = 5 \) and \( t = 8 \) is given in Figure 3.

**Example.** Examples of a super vertex-magic labeling of a kite graph with \( n = 4 \), \( t = 9 \) (\( t > n \)), and \( n = 8 \), \( t = 5 \) (\( t < n \)) are given in Figure 4.

**III. CONCLUSION**

According to result and discussion we found the reverse super vertex magic valuation of the \((n,t)\)-kite graph is \( k = \frac{n + t - 1}{2} \), for \( n \) is odd, \( t \) is even and \( n \) is even, \( t \) is odd for \( n \) odd where \( t \leq n \) and \( t > n \).

**REFERENCES**


