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On Bounds for Strong Metric Dimension of Two Families of Convex Polytopes

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Abstract

The strong metric dimension is a graph invariant that strengthens the classical notion of metric dimension by requiring that every pair of vertices be strongly resolved by at least one vertex lying on a shortest path between them. This parameter has attracted considerable attention due to its theoretical significance and applications in network analysis. Convex polytopes constitute an important class of highly symmetric planar graphs, yet their strong metric dimension has not been extensively studied. In this paper, we investigate the strong metric dimension of two families of convex polytopes. By analyzing their structural properties and distance relations, we determine bounds for the strong metric dimension for each family. Our results enrich the existing literature on metric-based graph invariants of polytope graphs and provide further insight into the interplay between geometric structure and strong resolvability.

Keywords: Resolving set, metric dimension, strong metric dimension.

1 Introduction

The concept of metric dimension of graphs, introduced independently by Slater [1] and Melter [2] in the 1970s, has attracted significant attention due to its wide range of applications in network navigation, chemistry, robotics, and combinatorial optimization. Kelenc et. al. [10] introduced the edge metric dimension $edim(G)$ of a connected graph $G = (V,E)$ as the minimum number of vertices needed so that every pair of distinct edges has a unique vector of distances to those vertices, using the distance from a vertex to an edge defined as the minimum distance to its endpoints. A related and more restrictive invariant, known as the strong metric dimension [3], was later introduced to strengthen the resolving condition by requiring that vertices lie on shortest paths. This parameter has proven useful in capturing finer structural properties of graphs and has been studied for various graph classes, including trees, product graphs, and planar graphs.

Convex polytopes form an important and well-structured class of graphs arising naturally from geometry and combinatorics. The graph of a convex polytope is highly symmetric and planar, making it a fertile ground for investigating graph invariants. In recent years, several researchers have examined the metric dimension and related parameters of families of convex polytopes, revealing interesting connections between their geometric structure and combinatorial properties. Javaid et. al. [5] studied the metric dimension of regular graphs, focusing on constructing infinite families where this parameter remains constant despite increasing graph order.

Chartrand et. al. [9] established bounds on metric dimension in terms of graph order and diameter, and completely characterizes connected graphs with extreme dimensions like 1, n-2 and n-1. A new proof for the metric dimension of trees is given, and this leads to sharp bounds for unicyclic graphs, showing how structural properties influence resolvability. The authors prove that for certain structured regular graphs (built via systematic constructions), a small fixed set of vertices uniquely determines all other vertices by distance, establishing constant metric dimension independent of size.

Imran et. al. [6] constructs infinite families of rotationally symmetric plane graphs whose metric dimension remains constant regardless of graph size. Using symmetry arguments and careful selection of resolving sets, they proved that a fixed small number of vertices uniquely determines all others by distance. Rafiullah et. al. [7] computed the exact metric

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dimensions for several families of convex polytope graphs by explicitly constructing minimal resolving sets. Their results extend resolvability analysis to geometric graph families, showing how structural properties of convex polytopes determine constant or size dependent metric dimension.

Raza et. al. [8] extends the concept of mixed metric dimension, the smallest set of vertices that distinguishes all vertices and edges by distance to certain rotationally symmetric graph families. For a rotationally symmetric family, they proved a constant mixed metric dimension of 5, illustrating that symmetry can bound this parameter regardless of graph size. Sharma et. al. [11] studied the edge resolving number of the pentagonal circular ladder graph, determining the smallest vertex set that uniquely distinguishes every pair of edges by distance. However, despite these advances, results concerning the strong metric dimension of convex polytopes remain relatively limited.

The pentagonal circular ladder is a well-known family of graphs obtained by arranging pentagons in a circular fashion such that consecutive pentagons share a common edge, forming a closed ladder-like structure. This graph can be viewed as a special case of a circular ladder graph in which each rung corresponds to a pentagonal face. Due to its regularity, planarity, and high degree of symmetry, the pentagonal circular ladder naturally arises in the study of polyhedral and chemical graph theory. From a structural point of view, the pentagonal circular ladder is closely related to the graphs of certain convex polytopes and fullerene-like structures. Its vertices and edges exhibit repetitive patterns that allow explicit computation of distances, diameters, and other metric properties.

These characteristics make the pentagonal circular ladder a suitable structure for investigating distance-based graph invariants. In recent years, distance-related parameters such as the metric dimension and strong metric dimension have gained increasing attention because of their applications in network navigation, combinatorial optimization, and pattern recognition. The pentagonal circular ladder, with its symmetric structure and predictable shortest-path behavior, provides an ideal framework for analyzing such invariants. Studying its strong metric dimension not only contributes to the theoretical understanding of this graph family but also offers insight into the role of symmetry in determining strong resolving sets.

In this paper, we study the strong metric dimension of two specific families of convex polytopes. By exploiting their structural regularities and shortest-path properties, we derive bounds of the strong metric dimension for each family. Our results extend existing work on metric-based parameters of polytope graphs and contribute to a deeper understanding of how geometric symmetry influences strong resolving sets. The techniques developed here may also be useful for analyzing other classes of planar and highly symmetric graphs.

2 The graph of pentagonal circular ladder

Let us denote the graph of pentagonal circular ladder by $PCL(n)$, where $n \geq 3$. The vertex set of $PCL(n)$ is $V(PCL(n)) = \{a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, c_1, c_2, ..., c_n\}$ and the edge set of $PCL(n)$ is $E(PCL(n)) = \{a_i a_{i+1} : 1 \leq i \leq n\} \cup \{a_i b_i : 1 \leq i \leq n\} \cup \{b_i c_i : 1 \leq i \leq n\} \cup \{c_i b_{i+1} : 1 \leq i \leq n\}$ where $a_{n+1} = a_1$ and $b_{n+1} = b_1$. The graph of $PCL(n)$ is shown in Figure 1.

There are exactly n pentagons in the graph of $PCL(n)$. It can been clearly that the number of elements in $V(PCL(n))$ is $3n$ and number of elements in $E(PCL(n))$ is $4n$. For each $n \geq 3$, we will get a convex polytope, so $PCL(n)$ for $n \geq 3$ represents a family of convex polytopes.

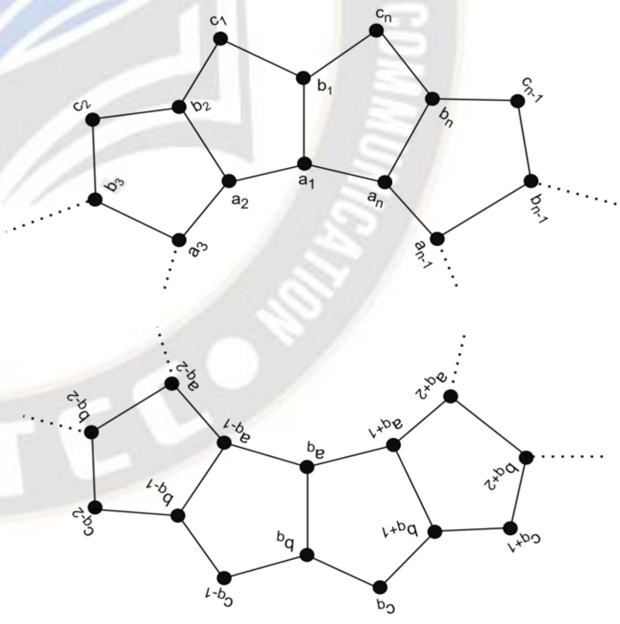


Figure 1: Graph of $PCL(n)$

3 The graph of convex polytope $PCL^*(n)$

If we join b_i with b_{i+1} where $1 \leq i \leq n$ and $b_{n+1} = b_1$ in the graph of $PCL(n)$, then we get another family of convex polytopes denoted by $PCL^*(n)$. The similarity between $PCL(n)$ and $PCL^*(n)$ is that $V(PCL(n)) = V(PCL^*(n))$. But the edge set of $PCL^*(n)$ contains n more number of edges than that in $PCL(n)$. In $PCL^*(n)$, the edge set is $E(PCL(n)) = \{a_ia_{i+1} : 1 \le i \le n\} \cup \{a_ib_i : 1 \le i \le n\} \cup \{b_ic_i : 1 \le i \le n\} \cup \{c_ib_{i+1} : 1 \le i \le n\} \cup \{b_ib_{i+1} : 1 \le i \le n\}$ where $a_{n+1} = a_1$ and $b_{n+1} = b_1$. The graph of $PCL^*(n)$ is shown in Figure 2.

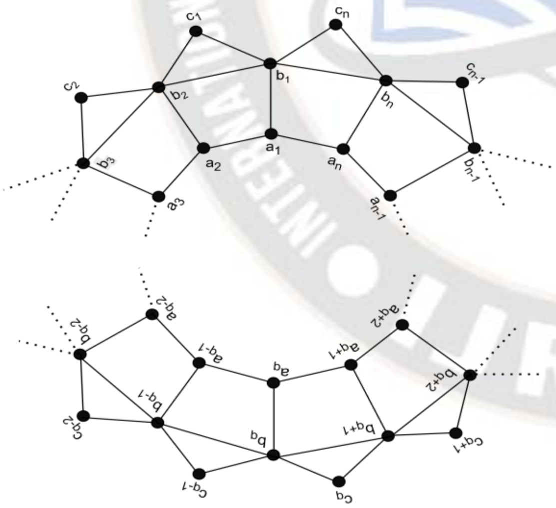
There are exactly n triangles in the graph of $PCL^*(n)$. It can been clearly that the number of elements in $V(PCL^*(n))$ is $3n$ and number of elements in $E(PCL^*(n))$ is $5n$. For each $n \ge 3$ in the graph of $PCL^*(n)$, we will get a convex polytope, so $PCL^*(n)$ for $n \ge 3$ represents a family of convex polytopes.

4 Strong Metric Dimension of Pentagonal Circular Ladder $PCL(n)$ and $PCL^*(n)$

In this section, we discuss about an upper bound for the strong metric dimension for the family of pentagonal circular ladder and for the family of convex polytopes $PCL^*(n)$, which is also derived from the family of pentagonal circular ladders.

Theorem 1. *The strong metric dimension of $PCL(n)$ is less than or equal to $2n$ i.e., $sdim(PCL(n)) \le n + \lceil \frac{n}{2} \rceil$, where $n > 6$ is odd positive integer.*

Proof. Let $S_R = \{a_1,a_3,a_5,\ldots,a_n,c_1,c_2,c_3,\ldots,c_n\}$ be a subset of vertex set of $PCL(n)$. Since, $n > 6$ is an odd positive integer, therefore $n = 2q + 1$ for some positive integer q. We shall show



that S_R is a strong resolving set.

For this, let a and b be two arbitrary vertices from $V(PCL(n)$. then there are three possibilies, that are, either $a,b \in S_R$ or $a \in S_R$ and $b \in T$ or $a,b \in T$. If $a,b \in S_R$ then this pair is resolved strongly by a as well as b. If $a \in S_R$ and $b \in T$, then this pair is resolved strongly by a. So, it only remains to show that every pair of vertices from the set $T = \{a_2,a_4,a_6,\ldots,a_{2q},b_1,b_2,b_3,\ldots,b_n\}$ is strongly resolved by some element of S_R.

Let $A = \{a_2,a_4,a_6,\ldots,a_{2q}\}$ and $B = \{b_1,b_2,b_3,\ldots,b_n\}$. Then $T = A \cup B$. If $x,y \in T$, then either $x,y \in A$ or $x,y \in B$ or $x, \in A$ and $y \in B$. So there are three cases:

Case I: When $x \in A$ and $y \in A$.

In this case the pair $x,y \in T$ is strongly resolved by some vertex among the vertices $a_1,a_3,\ldots,a_n$ as $a_1,a_2,\ldots,a_n$ forms a cycle and strong metric dimension of cycle having n vertices is $\lceil \frac{n}{2} \rceil$.

Case II: When $x \in B$ and $y \in B$.

In this case the pair $x,y \in T$ is strongly resolved by some vertex among the vertices $c_1,c_2,\ldots,c_n$ as $b_1,c_1,b_2,c_2,\ldots,b_n,c_n$, forms a cycle and strong metric dimension of cycle having n vertices is $\lceil \frac{n}{2} \rceil$.

Case III: When $x \in A$ and $y \in B$.

There are different possible pair of vertices such that one vertex is in A and other vertex is in B. Table 1 shows some pair of vertices from the set T such that one vertex is in A and other vertex is in B and the corresponding vertex from the set S_R that strongly resolves that pair.

Similarly, one can check for other pair of vertices from T that every pair of vertices from T is strongly resolved by some vertex of S_R.

Thus $sdim(PCL(n)) \le n + \lceil \frac{n}{2} \rceil$.

Table 1: Some pair of vertices from the set T such that one vertex is in A and other vertex is in B and the corresponding vertex from the set S_R that strongly resolves that pair

Some pair of vertices from T such that one vertex is in A and other vertex is in B	Corresponding vertex which strongly resolve that pair
$\{a_2,b_1\} \subset T$	cn
$\{a_2,b_2\} \subset T$	c_2
$\{a_2,b_3\} \subset T$	c_3
$\{a_2,b_r\} \subset T$; $4 \le r \le q + 2$	c_r
$\{a_2,b_r\} \subset T$; $q + 3 \le r \le 2q + 1$	$cr-1$
$\{a_4,b_1\} \subset T$	cn

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$\{a_4,b_2\}\subset T$	c_1
$\{a_4,b_3\}\subset T$	c_2
$\{a_4,b_4\}\subset T$	c^1
$\{a_4,b_r\}\subset T;\ 5\le r\le q+2$	$cr{+}1$
$\{a_4,b_r\}\subset T;\ q+3\le r\le 2q+1$	$cr{-}2$
$\{a_6,b_1\}\subset T$	cn
$\{a_6,b_2\}\subset T$	c_1
$\{a_6,b_3\}\subset T$	c_2
$\{a_6,b_4\}\subset T$	c_3
$\{a_6,b_5\}\subset T$	c_4
$\{a,b_r\}\subset T;\ 5\le r\le q+2$	$cr{+}2$
$\{a_4,b_r\}\subset T;\ q+3\le r\le 2q+1$	$cr{-}3$

□

Theorem 2. *The strong metric dimension of $PCL^*(n)$ is less than or equal to $n+\lceil\frac{n}{2}\rceil$, i.e., $sdim(PCL^*(n))\le n+\lceil\frac{n}{2}\rceil$, where $n>6$ is odd positive integer.*

Proof. Let $S_R=\{a_1,a_3,a_5,\ldots,a_n,c_1,c_2,c_3,\ldots,c_n\}$ be a subset of vertex set of $PCL^*(n)$. Since, $n>6$ is an odd positive integer, therefore $n=2q+1$ for some positive integer q. We shall show that S_R is a strong resolving set.

For this, let a and b be two arbitrary vertices from $V(PCL^*(n))$. then there are three possibilies, that are, either $a,b\in S_R$ or $a\in S_R$ and $b\in T$ or $a,b\in T$. If $a,b\in S_R$ then this pair is resolved strongly by a as well as b. If $a\in S_R$ and $b\in T$, then this pair is resolved strongly by a. So, it only remains to show that every pair of vertices from the set $T=\{a_2,a_4,a_6,\ldots,a_{2q},b_1,b_2,b_3,\ldots,b_n\}$ is strongly resolved by some element of S_R.

Let $A=\{a_2,a_4,a_6,\ldots,a_{2q}\}$ and $B=\{b_1,b_2,b_3,\ldots,b_n\}$. Then $T=A\cup B$. If $x,y\in T$, then either $x,y\in A$ or $x,y\in B$ or $x,\in A$ and $y\in B$. So there are three cases:

Case I: When $x\in A$ and $y\in A$.

In this case the pair $x,y\in T$ is strongly resolved by some vertex among the vertices $a_1,a_3,\ldots,a_n$ as $a_1,a_2,\ldots,a_n$ forms a cycle and strong metric dimension of cycle having n vertices is $\lceil\frac{n}{2}\rceil$.

Case II: When $x\in B$ and $y\in B$.

In this case the pair $x,y\in T$ is strongly resolved by some vertex among the vertices $c_1,c_2,\ldots,c_n$ as $b_1,c_1,b_2,c_2,\ldots,b_n,c_n$, forms a cycle and strong metric dimension of cycle having n vertices is $\lceil\frac{n}{2}\rceil$.

Case III: When $x\in A$ and $y\in B$.

There are different possible pair of vertices such that one vertex is in A and other vertex is in B.

From Case I and Case II, it can be see that $sdim(PCL(n))\le n+\lceil\frac{n}{2}\rceil$.

framework that may be extended to other structured graph families arising from geometric constructions. Future research may focus on determining exact strong metric dimensions for additional classes of convex polytopes, investigating asymptotic behavior for generalized families, or exploring algorithmic approaches for computing strong resolving sets in highly symmetric graphs.

Ethical Approval

There is no applicability of ethical approval.

Competing interests

There is no competing interests among the authors.

Authors' contributions

All authors' contributed equally in this work.

Availability of data and materials

The results of this study were not supported by any data or software, and none has been generated.

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[1] **Conclusion**

In this paper, we investigated the strong metric dimension of two families of convex polytopes and established upper bound for each class. By analyzing the structural properties and distance relationships inherent in these graph representations of convex polytopes, we derived explicit bounds. Our results demonstrate how symmetry, vertex arrangement, and cycle interactions within convex polytope graphs significantly influence strong resolving sets. The techniques developed in this work not only provide sharper bounds for the specific families considered, but also offer a systematic

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