Approximate Analytical Solution for Water Infiltration Problem in Unsaturated Soil

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***Abstract*—** In this paper, the approximate analytical solution of Richards Equation which is one of the most renowned equations to illustrate the behavior of the infiltration of unsaturated zones is obtained by the Variational Homotopy Perturbation Method (VHPM). Variational Homotopy Perturbation Method is blend of Variational iteration method and Homotopy perturbation method. The present method performs extremely well in terms of efficiency and simplicity.

Keywords- Richards Equation, Variational Homotopy Perturbation Method, Lagrange multiplier

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I. INTRODUCTION

Richards equation which is a nonlinear parabolic differential equation is modeled by flow in porous media In the recent past, the flow equations has been solved analytically after making some simplifying assumptions for simple boundary conditions for steady state isothermal flow through saturated isotropic homogeneous soils. The more complicated problems are unsteady flows through unsaturated porous media resulting in nonlinear partial differential equations. Richards applied the heat flow equation to unsaturated flow and assumed that the flow of water in unsaturated media obeys Darcy's law. He developed a general equation of flow in unsaturated media in which the water content and capillary conductivity are independent functions of the capillary potential. Various authors have considered the flow equation from different view point under specific initial and boundary conditions. Three standard form of the unsaturated flow equation is identified by the "h -based" i.e. pressure head based form, the "θ -based" i.e. moisture content based form and the "mixed" form with pressure and moisture content based.

In the present paper the unsaturated flow equation is identified by the θ based for the specific relationships considered which leads to a non-linear partial differential equation, a well known Richard's equation which is an excellent test problem for numerical method as well as analytical methods. In this chapter, an analytic technique, namely the Variational Homotopy Perturbation Method (VHPM) has been applied for solving Richards' equation which is one of the most familiar equations to illustrate the behavior of the infiltration of unsaturated zones in soil as a porous medium. Unlike most of the numerical and analytical techniques Variational Homotopy Perturbation Method is independent of Boundary conditions. The approximate analytical results are compared with the exact solution

II. MATHEMATICAL FORMULATION

Richards [1] derived a governing equation for water flow in soil based on continuum mechanics. In this model, the continuity equation was coupled with Darcy's law as a momentum equation. The following equation is known as the one-dimensional form of Richards' equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial z}\left(D\frac{\partial u}{\partial z}-K\right) \tag{1}$$

Where u is unsaturated soil moisture content, K is conductivity and D is soil water diffusivity. The sophistication of the analytical and numerical methods that are available to solve the governing equation of unsaturated flow in soils (Richards's equation) makes it necessary to have suitable models for parameters in the equation. Basically, there are three commonly used models The Brooks Corey model, the van Genuchten model, the exponential model. The commonly used model, Brook-Corey model introduces a well defined air entry value that is associated with the largest pore size, assuming complete wet ability, Brook-Corey can be simplified to the following equations by some further considerations [8,9],

$$K(u)=K_0u^k \quad \text{for } m\geq 1 \tag{2}$$

$$D(u)=D_0(n+1)u^n \tag{3}$$

where K_0, D_0, n and k are constants representing soil properties such as pore-size distribution, particle shape, etc. In these relations, u is scaled between 0 and 1 and the form of diffusivity is normalized so that

$$\int_0^1 D(u)d(u)=1 \qquad \forall n$$

Most of the results in this chapter can be applied to the general values of k and n.

However, two cases for conductivity exponents (with equal soil water diffusivity) and the solutions were tried. The hyperbolic tangent function is commonly applied to solve these transform equations [1]. The general form of Burgers equation in the order of (n,1) is

$$u_t+a(u^n)_x+bu_{xx}=0 \qquad n\geq 1, a, b\neq 1 \tag{4}$$

and its exact solution [1] is

$$u(x,t)=\left\{\frac{c}{2a}\left[1+\tanh\left(\frac{c(n-1)}{2b}(x-ct)\right)\right]\right\}^{\frac{1}{n-1}} \tag{5}$$

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III. VARIATIONAL HOMOTOPY PERTURBATION METHOD

To express the fundamental proposal of the Variational homotopy method[2,3,4,5], we consider general differential equation

$$Lu + Nu = g(x) \tag{6}$$

Where L is a linear operator, N is a nonlinear operator, and g(x) is the forcing term. According to variational iteration method, we can construct a correct functional [6]as follows:

$$u_{n+1}(x) = u_{n(x)} + \int_0^t \lambda(\tau)\left(Lu_n(\tau) + Nu_n(\tau) - g(\tau)\right)d\tau \tag{7}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. We apply restricted variations to nonlinear term Nu so that we can determine the multiplier. Making the above functional stationary, noticing that $\delta\tilde{u}_k = 0$, In the homotopy perturbation method[7], the basic assumption is that the solutions can be written as a power series in *p*. The variational homotopy perturbation method is obtained by the elegant coupling of correction functional of variational iteration method with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)}u_n = u_{0(x)} + p\int_0^t \lambda(\tau)\left(\sum_{n=0}^{\infty} p^{(n)}L(u_n(\tau)) + N\sum_{n=0}^{\infty} p^{(n)}u_n(\tau)\right)d\tau - \int_0^t \lambda(\tau)g(\tau)\,d\tau \tag{8}$$

A comparison of like powers of p gives solutions of various orders. This method does not resort to linearization or assumptions of weak nonlinearity, the solutions generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

IV. APPROXIMATE ANALYTICAL SOLUTION OF RICHARDS EQUATION

Case 1: In this case a conductivity term is selected as a function of cubic water content and constant

$$K = \frac{u^3}{3}\,cm/h \quad \text{And} \quad D = 1cm^{2/h},$$

Therefore the Richards Equation becomes

$$u_t + u^2u_x - u_{xx} = 0 \tag{9}$$

With the initial condition [1]

$$u(x,0) = \sqrt{\frac{1}{2}\left(1 + \tanh\left(\frac{-x}{3}\right)\right)} \tag{10}$$

According to Variational Homotopy Perturbation method, we construct the correction functional for equation (9) as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^\tau \left[\lambda(\tau)\left(\frac{\partial u_n}{\partial t} + u^2\frac{\partial u_n}{\partial x} - \frac{\partial^2 u_n}{\partial x^2}\right)\right]d\tau$$

which yields the stationary conditions

$$\lambda'(\tau) = 0$$
$$1 + \lambda(\tau) = 0$$

Therefore, the general Lagrange multiplier can be readily identified as $\lambda = -1$, which yields the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^\tau \left(\frac{\partial u_n}{\partial t} + u^2\frac{\partial u_n}{\partial x} - \frac{\partial^2 u_n}{\partial x^2}\right)d\tau \tag{11}$$

Applying the variational homotopy perturbation method, we get

$$u_0 + pu_1 + p^2u_2 + \ldots = f(x) - p\int_0^t \left[\begin{matrix}\left(u_0 + pu_1 + p^2u_2 + \ldots\right)^2 \\ \frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \ldots\right)\end{matrix}\right]d\tau + p\int_0^t \frac{\partial^2}{\partial x^2}\left(u_0 + pu_1 + p^2u_2 + \ldots\right)d\tau \tag{12}$$

Comparing the coefficient of like powers of *p*, we have

$$p^0 : u_0(x,t) = \sqrt{\frac{1}{2}\left(1 + \tanh\left(\frac{-x}{3}\right)\right)}$$

$$p^1 : u_1(x,t) = \frac{tsech\left[\frac{x}{3}\right]^2}{18\sqrt{2 - 2\tanh\left[\frac{x}{3}\right]}}$$

$$p^2 : u_2(x,t) = \frac{t^2sech\left[\frac{x}{3}\right]^4\left(\cosh\left[\frac{x}{3}\right] - 3\sinh\left[\frac{x}{3}\right]\right)\left(\cosh\left[\frac{x}{3}\right] - \sinh\left[\frac{x}{3}\right]\right)}{432\sqrt{2 - 2\tanh\left[\frac{x}{3}\right]}} + \frac{t^2sech\left[\frac{x}{3}\right]^4\left(-10 + 5\cosh\left[\frac{2x}{3}\right] - 3\sinh\left[\frac{2x}{3}\right]\right)}{1296\sqrt{2 - 2\tanh\left[\frac{x}{3}\right]}} - \frac{t^2sech\left[\frac{x}{3}\right]^4}{216\sqrt{2 - 2\tanh\left[\frac{x}{3}\right]}}$$

$$p^2 : u_2(x,t) = \frac{1}{162}e^{2x/3}\left(-2 + e^{2x/3}\right)\left(\frac{1}{1 + e^{2x/3}}\right)^{5/2}t^2$$

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Similarly, further approximations can be obtained up to desired accuracy. The solution becomes

$$u(x,t)=\sqrt{\frac{1}{2}\left(1+\tanh\left(\frac{-x}{3}\right)\right)}+\frac{tsech\left[\frac{x}{3}\right]^2}{18\sqrt{2-2\tanh\left[\frac{x}{3}\right]}}$$
$$-\frac{1}{162}e^{2x/3}\left(-2+e^{2x/3}\right)\left(\frac{1}{1+e^{2x/3}}\right)^{5/2}t^2+\ldots \quad (13)$$

Case 2: In this case a conductivity term is selected as a function of quadric water content and constant

$$K=\frac{u^4}{4}\,cm/h \quad (14)$$

And $D=1cm^{2/h}$ Therefore the Richards Equation becomes

$$u_t+u^3u_x-u_{xx}=0 \quad (15)$$

With the initial condition [1]

$$u(x,0)=\sqrt[3]{\frac{1}{2}\left(1+\tanh\left(\frac{-3x}{8}\right)\right)} \quad (16)$$

According to Variational Homotopy Perturbation method, we construct the correction functional similar to case 1for equation (15) as

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^\tau\left(\frac{\partial u_n}{\partial t}+u^3\frac{\partial u_n}{\partial x}-\frac{\partial^2 u_n}{\partial x^2}\right)d\tau$$

Applying the variational homotopy perturbation method and comparing the coefficient of like powers of *p*, we have

$$p^0:u_0(x,t)=\sqrt{\frac{1}{2}\left(1+\tanh\left(\frac{-3x}{8}\right)\right)}$$

$$p^1:u_1(x,t)=-\frac{tsech\left[\frac{3x}{8}\right]^3\left(-\cosh\left[\frac{3x}{8}\right]+\sinh\left[\frac{3x}{8}\right]\right)}{32\times 2^{1/3}\left(1-\tanh\left[\frac{3x}{8}\right]\right)^{5/3}}$$

$$p^2:u_2(x,t)=\frac{1}{512}e^{3x/4}\left(\frac{1}{1+e^{3x/4}}\right)^{10/3}\left(-3-14e^{3x/4}+25e^{3x/2}\right)t^2$$

Similarly, further approximations can be obtained up to desired accuracy. The solution becomes

$$u(x,t)=\sqrt[3]{\frac{1}{2}\left(1+\tanh\left[\frac{-3x}{8}\right]\right)}-\frac{tSech\left[\frac{3x}{8}\right]^3\left(-\cosh\left[\frac{3x}{8}\right]+\sinh\left[\frac{3x}{8}\right]\right)}{322^{1/3}\left(1-\tanh\left[\frac{3x}{8}\right]\right)^{5/3}}$$
$$+\frac{1}{512}e^{3x/4}\left(\frac{1}{1+e^{3x/4}}\right)^{10/3}\left(-3-14e^{3x/4}+25e^{3x/2}\right)t^2+\ldots \quad (17)$$

V. RESULTS AND DISCUSSION

Here an approximate solution is obtained for two cases of conductivity exponents and compared with the exact solution to put emphasis on the accuracy of the present method with equal soil water diffusivity. The solutions obtained of case 1 and case 2 are compared with the exact solution at particular time has been shown in Table 1 and Table 2 respectively which shows that solution are in good harmony with exact solution.The plots of the numerical solution obtained for various values of time and space, considering a=1/3, b=1, n=3 and c=1/3 is shown in Fig.1(A) and in Figure 1(B) solution for case 2 is represented considering a=1/4, b= -1, n=4 and c=1/4.In both the figures the moisture content decreases as distance increases and slightly increases with time which is physically consistent with the real phenomena. The solution expressed by (13) & (17) show the approximate analytical solution of moisture content at any distance for t > 0.

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TABLE 1 *Comparison of VHPM solutions of (Case 1) with Exact solutions at a=1/3, b=1, n=3 and c=1/3 at different times*

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	x	Exact Solution	VHPM Solution
t=0.1	-10	0.99981912856359	0.99981912829109
	-5	0.99242266669203	0.99242254480559
	5	0.286034335847339	0.286157154040
	10	0.082584125848703	0.082622518412632
t=0.2	-10	0.99982248711434	0.9998224852278
	-5	0.9925613344816	0.99256081692331
	5	0.2877851655633094	0.288276413397977
	10	0.0832551546705483	0.08325515467054
t=0.3	-10	0.9998257833228529	0.999825777302473
	-5	0.9926975018610925	0.992696270377752
	5	0.2895459227148223	0.289651174486001
	10	0.0836223063621502	0.083667779665323
t=0.4	-10	0.9998290183455204	0.999829004515097
	-5	0.992831212571408	0.992838905168899
	5	0.2913166441059236	0.293181437304531
	10	0.0841462720073618	0.084160393396957
t=0.5	-10	0.9998321933173414	0.999832166865680
	-5	0.9929625096368975	0.992958721296760
	5	0.2930973661762994	0.293067201853569
	10	0.0846735148391933	0.084632995865450

TABLE 2 Comparison of VHPM solutions of (Problem 1) with Exact solutions at a=1/4, b= - 1, n=4 and c=1/4 at different times.

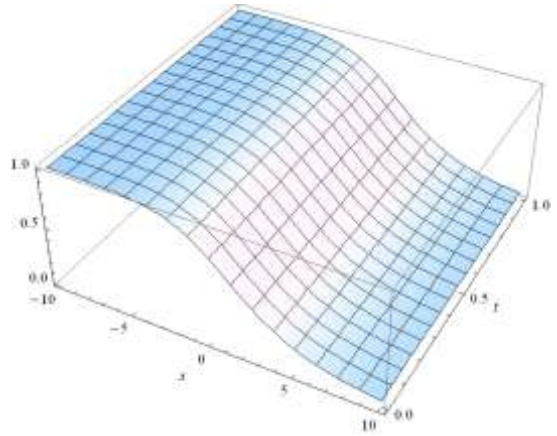

Figure 1: (A) Graphical representation of x for different values of time by VHPM (case 1)

	x	Exact Solution	VHPM Solution
t=0.1	-10	0.99937824786166	0.99937824671303
	-5	0.98299862120683	0.9829985950282
	5	0.18759523876262	0.1875952140202
	10	0.0360491348248	0.0360491267868
t=0.2	-10	0.99939189972632	0.99939189058762
	-5	0.98336299053799	0.98336278201703
	5	0.189616293992895	0.18961609623752
	10	0.03645138206566	0.0364513175890
t=0.3	-10	0.99940525210338	0.99940522142847
	-5	0.98371974428774	0.9837190435801
	5	0.191657482692509	0.191656815919773
	10	0.036858105627242	0.036857887432469
t=0.4	-10	0.99941831155124	0.99941823923561
	-5	0.98406903345595	0.98406737971767
	5	0.19371895194698	0.19371737306695
	10	0.03726935491456	0.03726883631715
t=0.5	-10	0.99943108448515	0.99943094400902
	-5	0.98441100639318	0.98440779042947
	5	0.195800848122166	0.195797767679069
	10	0.037685179864062	0.037684164243098

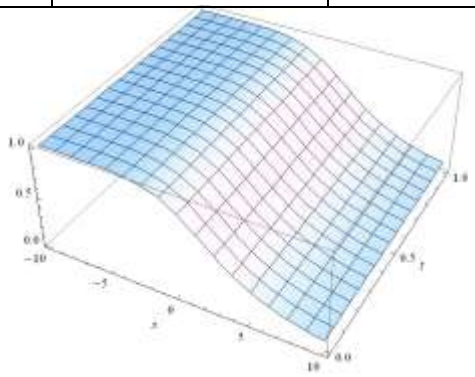

Figure 1: (B): Graphical representation of x for different values of time by VHPM (case 2)

VI. CONCLUSION

In this paper, solution of Richards equation by Brokes-Corey Model is obtained by applying Variational Homotopy Perturbation method with specific initial conditions. On comparing the obtained results using Variational Homotopy perturbation method with the available exact solution, it is concluded that the present method is easy, accurate, and efficient with several advantages over other traditional analytical methods. The method can be equally well applied to the physical problems in various phenomena in sciences and engineering.

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