Study of Diverging-Converging Nozzle based on Explicit Mac Cormack's Method

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Abstract— We present Numerical study of an explicit finite-difference scheme for simulation of fluid flow through nozzle. Equations for quasi-one-dimensional flow were used which were derived from Continuity, Momentum and Energy principles. Effect of Courant number on supersonic flow through nozzle has also been investigated. The non-dimensional numbers such as density ratio, temperature ratio and Mach number were quantified both by analytical and numerical methods. The numerical solution was done using Mac-Cormack's predictor-corrector technique and coded in MATLAB. The method used is a simple and computer efficient technique which gives second order accuracy both in time and space.

Keywords-*component; Explicit ;MacCormack ;Courant;Mach Number;Converging-Diverging Nozzle*

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I. INTRODUCTION

Direct numerical stimulation of fluid flow through nozzle has always been a challenging problem and an area of immense interest for modern engineers. Converging-Diverging nozzle has been extensively used in a wide range of Industrial applications ranging from steam turbines to rocket engine nozzles.

The condition for operation so that the nozzle produces supersonic flow is that the Mach number at throat should be equal to one. This condition is known as choked flow. As the fluid expands further in the diverging section, the fluid flow increases to supersonic velocities.(Mach Number>1).If the choke condition is not met, the nozzle simply acts as a Venturi tube and would produce subsonic flow even at the outlet.

This paper uses an explicit finite difference scheme for numerical solution of fluid flow. This scheme is fully explicit and second order accurate in both time and space.

Anderson [3] used finite differencing method to discretize the basic equations. The resulting set of non-linear equations is then solved with Mac-Cormack's scheme. It solves the equation by time integration starting from an estimated initial state until the solution stabilizes. J. Prins [10] has compared and validated various approaches for quasi-one-dimensional steady state models for gas leakage. A. Alakshi [11] discussed the implementation of finite volume method for nozzle flow and shock capturing using this approach.

This paper first discusses the details of Matlab Code as a flow solver. Mac-Cormack's technique is a variation of Lax-Wendroff Approach in which the latter's complex second order derivative is replaced by forward differences on the predictor and rearward differences on the corrector.

Mac-Cormack's scheme is perfectly satisfactory and the results have been known to approximately correct for many fluid flow applications. We also obtain similar results even if we used rearward differences on predictor and forward difference on the corrector. This scheme is well suited for non-linear equations as frequently encountered in fluid flow applications.

II. METHODOLOGY

A. Governing Equations

The fluid flow through the nozzle is governed by continuity, momentum and energy equations. Since the flow is assumed to be quasi-One dimensional flows, the conventional equations are slightly modified and Non-Conservation equations as suitable to unsteady, quasi-one-dimensional flow:

Continuity Equation:

$$\frac{\partial(\rho A)}{\partial t}+\rho A\frac{\partial V}{\partial x}+\rho V\frac{\partial A}{\partial x}+VA\frac{\partial \rho}{\partial x}=0 \qquad (1)$$

Momentum Equation:

$$\rho\frac{\partial V}{\partial t}+\rho V\frac{\partial V}{\partial x}=-R(\rho\frac{\partial T}{\partial x}+T\frac{\partial \rho}{\partial x}) \qquad (2)$$

Energy Equation:

$$\rho c_v\frac{\partial T}{\partial t}+\rho V c_v\frac{\partial T}{\partial x}=-\rho RT[\frac{\partial V}{\partial x}+V\frac{\partial(\ln A)}{\partial x}] \qquad (3)$$

B. Non Dimensional Numbers

In most the cases, it's more convenient to express all the parameters in non-dimensional format so that it is easily presented in terms of a ratio whose value varies from 0 to 1 and to present the obtained results. The following non-dimensional parameters have been used throughout the paper.

$$T' = \frac{T}{T_0}; \rho' = \frac{\rho}{\rho_0}; x' = \frac{x}{L}; a_0 = \sqrt{\gamma R T_0};$$

$$V' = \frac{V}{a_0}; t' = \frac{t}{L/a_0}; A' = \frac{A}{A^*}$$

The governing equations are modified using Non-Dimensional Numbers as established above.

Continuity Equation:

$$\frac{\partial \rho'}{\partial t'} = -\rho' \frac{\partial V'}{\partial x'} - \rho' V' \frac{\partial (\ln A')}{\partial x} - V' \frac{\partial \rho'}{\partial x'} \quad (4)$$

Momentum Equation:

$$\frac{\partial V}{\partial t} = -V \frac{\partial V}{\partial x} - \frac{1}{\gamma}\left(\frac{\partial T'}{\partial x'} + \frac{T'}{\rho'}\frac{\partial \rho'}{\partial x'}\right) \quad (5)$$

Energy Equation:

$$\frac{\partial T'}{\partial t'} = -V' \frac{\partial T'}{\partial x'} - (\gamma - 1) T' \left[\frac{\partial V'}{\partial x'} + V' \frac{\partial (\ln A')}{\partial x'}\right] \quad (6)$$

C. Fluid Flow Variable Values

The geometry of the nozzle has been assumed and the following variations with respect to distance 'x' from the nozzle inlet have been used. Since the flow is assumed to quasi-one-dimensional flow, the properties at a given section are uniform.

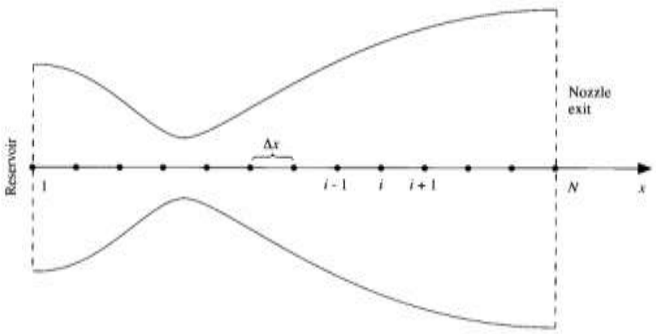

Figure 1 Nozzle Geometry

$$A = 1 + 2.2(x - 1.5)^2$$
$$\rho = 1 - 0.3146x$$
$$T = 1 - 0.2314x \qquad 0 \le x \le 3$$
$$V = (0.1 + 1.09x)T^{0.5}$$

D. Mac Cormack Scheme

It's a two step Predictor-Corrector formula as shown below:
The Predictor formula has been implemented using forward difference.

$$\left(\frac{\partial \rho'}{\partial t'}\right)_i^t = -\rho' \frac{\partial (V_{i+1}^t - V_i^t)'}{\partial x'} - \rho' V' \frac{\partial (\ln A_{i+1}^t - A_i^t)'}{\partial x} - V' \frac{\partial (\rho_{i+1}^t - \rho_i^t)'}{\partial x'} \quad (7)$$

$$\left(\frac{\partial V}{\partial t}\right)_i^t = -V \frac{\partial (V_{i+1}^t - V_i^t)'}{\partial x} - \frac{1}{\gamma}\left(\frac{\partial (T_{i+1}^t - T_i^t)'}{\partial x'} + \frac{T'}{\rho'}\frac{\partial (\rho_{i+1}^t - \rho_i^t)'}{\partial x'}\right) \quad (8)$$

$$\left(\frac{\partial T'}{\partial t'}\right)_i^t = -V' \frac{\partial (T_{i+1}^t - T_i^t)'}{\partial x'} - (\gamma - 1) T' \left[\frac{\partial (V_{i+1}^t - V_i^t)'}{\partial x'} + V' \frac{\partial (\ln A_{i+1}^t - A_i^t)'}{\partial x'}\right] \quad (9)$$

Using Euler's Method first order time derivatives can be obtained by

$$\rho'^{t+\Delta t}_i = \rho'^t_i + \left(\frac{\partial \rho'}{\partial t'}\right)_i^t \Delta t$$

$$V'^{t+\Delta t}_i = V'^t_i + \left(\frac{\partial V'}{\partial t'}\right)_i^t \Delta t$$

$$T'^{t+\Delta t}_i = T'^t_i + \left(\frac{\partial T'}{\partial t'}\right)_i^t \Delta t \quad (10)$$

Corrector Formula

This is implemented using rearward difference method.

$$\left(\frac{\partial \rho'}{\partial t'}\right)_i^{t+\Delta t} = -\rho' \frac{\partial (V_i^{t+\Delta t} - V_{i-1}^t)'}{\partial x'} - \rho' V' \frac{\partial (\ln A_i^{t+\Delta t} - A_{i-1}^t)'}{\partial x} - V' \frac{\partial (\rho_i^{t+\Delta t} - \rho_{i-1}^t)'}{\partial x'} \quad (11)$$

$$\left(\frac{\partial V}{\partial t}\right)_i^{t+\Delta t} = -V \frac{\partial (V_i^{t+\Delta t} - V_{i-1}^t)'}{\partial x} - \frac{1}{\gamma}\left(\frac{\partial (T_i^{t+\Delta t} - T_{i-1}^t)'}{\partial x'} + \frac{T'}{\rho'}\frac{\partial (\rho_i^{t+\Delta t} - \rho_{i-1}^t)'}{\partial x'}\right) \quad (12)$$

$$\left(\frac{\partial T'}{\partial t'}\right)_i^{t+\Delta t} = -V'\frac{\partial(T_i^{t+\Delta t}-T_{i-1}^{t})'}{\partial x'} - (\gamma-1)T'\left[\frac{\partial(V_i^{t+\Delta t}-V_{i-1}^{t})'}{\partial x'} + V'\frac{\partial(\ln A_i^{t+\Delta t}-A_{i-1}^{t})'}{\partial x'}\right] \quad (13)$$

Average values are used for Corrected derivative.

$$\left(\frac{\partial \rho'}{\partial t'}\right)_{av} = \left(\frac{\partial \rho'}{\partial t'}\right)_i^t + \left(\frac{\partial \rho}{\partial t}\right)_i^{t+\Delta t} \quad (14)$$

$$\left(\frac{\partial V}{\partial t}\right)_{av} = \left(\frac{\partial V}{\partial t}\right)_i^t + \left(\frac{\partial V}{\partial t}\right)_i^{t+\Delta t} \quad (15)$$

$$\left(\frac{\partial T'}{\partial t'}\right)_{av} = \left(\frac{\partial T'}{\partial t'}\right)_i^t + \left(\frac{\partial T'}{\partial t'}\right)_i^{t+\Delta t} \quad (16)$$

Finally the corrected flow variables are given by,

$$\rho'^{t+\Delta t}_i = \rho'^t_i + \left(\frac{\partial \rho'}{\partial t'}\right)_{av}\Delta t \quad (17)$$

$$V'^{t+\Delta t}_i = V'^t_i + \left(\frac{\partial V'}{\partial t'}\right)_{av}\Delta t \quad (18)$$

$$T'^{t+\Delta t}_i = T'^t_i + \left(\frac{\partial T'}{\partial t'}\right)_{av}\Delta t \quad (19)$$

Time step is determined using the relation,

$$(\Delta t)_i^t = C\frac{\Delta x}{a_i^t + V_i^t}$$

$$(\Delta t)=\min\left((\Delta t)_1^t, (\Delta t)_2^t, \ldots.. (\Delta t)_N^t\right)$$

The time step chosen is the minimum of time steps as obtained at each node.

E. Nomenclature

A Cross-sectional area	Subscripts
L Length	N Total number of grid
P Pressure	i Station location
a Speed of sound	Superscripts
t Time	‘ Non-dimensional variable
V Velocity	∗ Throat location
x Length	
γ Ratio of specific heat	
ρ Density	

F. Bounday Conditions

By Linear Extrapolation, we obtain the values at the end nodes, At the Subsonic Inlet, we have:

$$V_1 = 2V_3 - V_3$$

$$\rho_1 = 1$$

$$T_1 = 1$$

This is independent of time, if we assume the reservoir to be ideal.

At the Supersonic Outlet, we have:

$$V_N = 2V_{N-1} - V_{N-2}$$

$$\rho_N = 2\rho_{N-1} - \rho_{N-2}$$

$$T_N = 2T_{N-1} - T_{N-2}$$

III. RESULTS

The following important results have been plotted after running the code successfully for 1400 time steps.

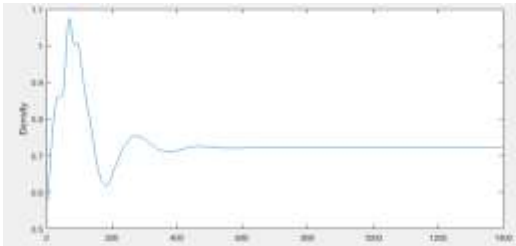

Figure 2: Non-Dimensional Density vs No of Iterations

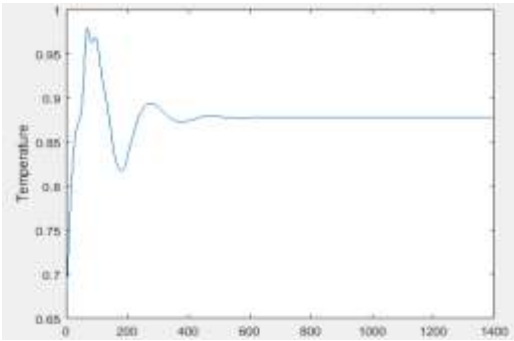

Figure 3: Non-Dimensional Temperature vs No of Iterations

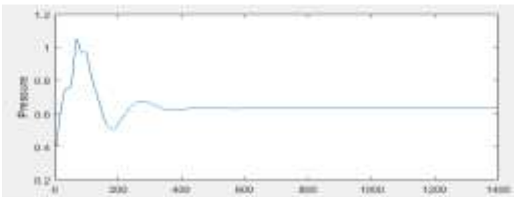

Figure 4: Non-Dimensional Pressure vs No of Iterations

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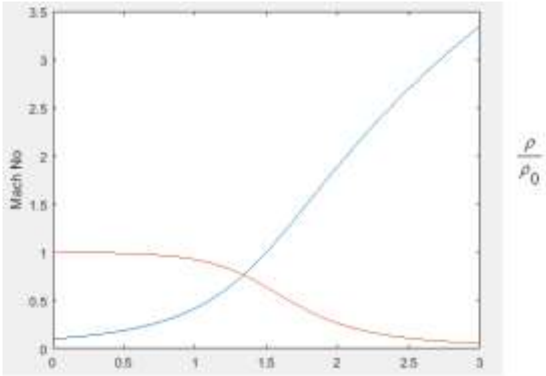

Figure 5: Non-Dimensional Distance vs Mach No & Non-Dimensional Density

Table 1: Non Dimensional Flow variables

Sr No.	$\frac{x}{L}$	$\frac{A}{A^*}$	$\frac{\rho}{\rho o}$ Numerical	$\frac{\rho}{\rho o}$ Exact	% difference	Mach No	Mach No Exact	% difference
1	0	5.95	1.000	0.995	0.500	0.099	0.099	0.134
2	0.1	5.312	0.998	0.994	0.355	0.112	0.112	0.170
3	0.2	4.718	0.997	0.992	0.500	0.125	0.125	0.170
4	0.3	4.168	0.994	0.990	0.412	0.143	0.143	-0.117
5	0.4	3.662	0.992	0.987	0.459	0.162	0.163	-0.310
6	0.5	3.2	0.987	0.983	0.424	0.187	0.187	0.029
7	0.6	2.782	0.982	0.978	0.377	0.216	0.216	0.101
8	0.7	2.408	0.974	0.970	0.374	0.252	0.252	0.045
9	0.8	2.078	0.962	0.958	0.467	0.296	0.296	-0.064
10	0.9	1.792	0.947	0.942	0.479	0.350	0.35	-0.116
11	1	1.55	0.924	0.920	0.434	0.416	0.416	-0.116
12	1.1	1.352	0.892	0.888	0.490	0.496	0.496	0.035
13	1.2	1.198	0.849	0.844	0.595	0.594	0.594	-0.029
14	1.3	1.088	0.792	0.787	0.625	0.710	0.71	0.037
15	1.4	1.022	0.721	0.716	0.674	0.846	0.846	-0.011
16	1.5	1	0.639	0.634	0.718	0.999	0.999	0.039
17	1.6	1.022	0.551	0.547	0.713	1.167	1.167	0.032
18	1.7	1.088	0.465	0.461	0.838	1.345	1.345	0.009
19	1.8	1.198	0.386	0.382	1.107	1.528	1.528	-0.026
20	1.9	1.352	0.318	0.315	1.042	1.710	1.71	0.020
21	2	1.55	0.262	0.258	1.445	1.890	1.89	0.005
22	2.1	1.792	0.216	0.213	1.332	2.065	2.065	-0.018
23	2.2	2.078	0.179	0.176	1.697	2.233	2.233	-0.002
24	2.3	2.408	0.150	0.147	1.755	2.394	2.394	0.009
25	2.4	2.782	0.126	0.124	1.647	2.549	2.549	0.000
26	2.5	3.2	0.107	0.105	2.046	2.696	2.696	0.011
27	2.6	3.662	0.092	0.090	2.010	2.839	2.839	-0.013
28	2.7	4.168	0.079	0.078	1.845	2.972	2.972	0.015
29	2.8	4.718	0.069	0.068	1.496	3.105	3.105	0.009
30	2.9	5.312	0.061	0.059	3.194	3.225	3.225	-0.004
31	3	5.95	0.053	0.052	1.626	3.353	3.353	0.005

As expected,the expanding flow has lesser pressure which translates into increase in velocity(Bernoulli's Equation) and hence the Mach Number increases.We have sonic flow at the throat(M=1) and supersonic flow in the divergent section.
Effect of Courant Number on the results have been summarised in the following table:

Table 2: Courant Number vs Non-Dimensional entities

Sr No.	$\frac{x}{L}$	$\frac{A}{A^*}$	$\frac{\rho}{\rho o}$	T/T0	P/P0	Mach No Exact
0.5	1.5	1	0.638580	0.836410	0.534110	0.999390
0.7	1.5	1	0.638929	0.836532	0.534484	0.999161
0.9	1.5	1	0.639285	0.836659	0.534864	0.998946
1	1.5	1	0.639466	0.836722	0.535056	0.998842
1.1	1.5	1	0.63965	0.836786	0.53525	0.998742
1.2	1.5	1	NaN	NaN	NaN	NaN

IV. CONCLUSION

We have investigated a quasi-one-dimensional flow through a nozzle having subsonic conditions at the inlet and supersonic conditions at the outlet. The present numerical solver scheme for nozzle flow has been developed with second order accuracy. The results obtained have been plotted and validate the results from Anderson [4]. Further the effect of Courant Number on the flow has also been studied. The results obtained using the numerical solution was in close agreement with the exact solution. The theoretical expectation of Courant Number less than one is a good estimate for converging solution for the numerical method being validated. Although we tested our Matlab solver up to a CFL no 1.12 and the code was found to be stable.

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