International Journal on Recent and Innovation Trends in Computing and Communication
Volume: 4 Issue: 4

ISSN: 2321-8169
934 - 939

Linear Approximation Approach to Two-Person Zero-Sum Game Using Symmetric Epsilon Fuzzy Payoff Matrix

M. S. Bapat
Department of Mathematics
Willingdon College, Sangli – 416 415, Maharashtra, India
Email: msbapat@gmail.com

S. J. Aher
Department of Basic Sciences
ADCET, Ashta - 416 301, Maharashtra, India
Email: sujataaher@gmail.com

S. N. Yadav
Department of Basic Sciences
ADCET, Ashta - 416 301, Maharashtra, India
Email: snyadavmath@gmail.com

M. D. Khedekar
Department of Basic Sciences
ADCET, Ashta - 416 301, Maharashtra, India
Email: khedekarmd@gmail.com

Abstract:- In this note Two-Person Zero-Sum Game with triangular epsilon fuzzy payoff matrix is discussed and obtained the optimal fuzzy strategies by using MAX-MIN approximation. We establish the Criterion of Fuzzy Minimax-Maximin by defining the ordering of epsilon fuzzy numbers. Results are exemplified by examples.

***Key words**: Epsilon fuzzy number; Fuzzy payoff matrix; Fuzzy saddle point; Fuzzy Two- Person Zero-Sum game, MAX-MIN approximation.*

*****

1. Introduction

Game theory is a one of the types of decision theory in which player's action is imprecise after taking into account all possible actions available to an opponent playing the same game. Fuzzy games are intended to model conflict situation with imprecise information, Payoffs strategies etc. The mathematical analysis of competitive problems is based upon 'Minimax-Maximin Criterion' of J. Von Neuman. The simplest type of competitive situations is two-person zero-sum game [4-6]. These games involve only two players; they are called *zero-sum* games because one player wins whatever the other player loses [7]. In this paper, we have proved 'Fuzzy Minimax-Maximin criterion' by using triangular epsilon fuzzy number [1] as cell entries of payoff matrix. We use MAX-MIN approximation [2] for establishing the Minimax-Maximin criterion.

In section 2 we give basic notions which are used in the sequel. In section 3 fuzzy Two- Person Zero-Sum game is discussed. In section 4 results are elaborated by empirical examples.

2. Preliminaries

Terminology of Two-Person Zero-Sum Games

- A two-person game is characterized by the strategies of each player and the payoff matrix.
- The payoff matrix shows the gain (positive or negative) for player 1 that would result from each combination of strategies for the two players. *Note that the matrix for player 2 is the negative of the matrix for player 1 in a zero-sum game.*
- The entries in the payoff matrix can be in any units as long as they represent the *utility (or value)* to the player.
- There are two key assumptions about the behavior of the players. The first is that both players are *rational*. The second is that both players are *greedy* meaning that they choose their strategies in their own interest (to promote their own wealth).
- The definition of a two-person zero-sum game in normal form amounts to defining sets of strategies A and B of players I and II respectively, and of the pay-off function H of player-I, defined on the set $A\times B$ of all situations (the pay-off function of player II is $-H$ by definition). Formally, a two-person zero-sum game Γ is given by a triplet $\Gamma=\langle A, B, H\rangle$. Play consists in the players choosing their strategies $a \in A$, $b \in B$, after which player I obtains the sum $H(a, b)$ from player II. Such a definition of a two-person zero-sum game is sufficiently general to include all variants of two-person zero-sum

games, including dynamic games and positional games provided that the sets of strategies and the pay-off function are properly described. A rational choice of actions (strategies) of the players in the course of a two-person zero-sum game is based on a minimax principle:

If *max a∈A, inf b∈B, H (a, b) =min b∈B, sup a ∈AH (a, b)* (1)

or

sup a ∈ *A*, inf *b* ∈ *B*, *H* (*a*, *b*)= inf *b* ∈*B*, *sup a* ∈*A H* (*a*, *b*) (2)

the game Γ has optimal strategies (ϵ-optimal strategies, respectively) for both players. The common value of both parts of equation (2) is called the value of the game Γ. However, equations (1) or (2) may not be valid even in the simplest cases.

For example. In a matrix game with payoff matrix

Player B

Player A $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The following equalities are valid:

$$\max_i \min_j a_{ij} = -1, \min_j \max_i a_{ij} = 1$$

Definition 1. [8] A *fuzzy number A* is a subset of the real line $\mathbb{R}$ with membership function $A:\mathbb{R} \to [0,1]$ such that *A* is normal, *A* is fuzzy convex and upper semi-continuous i.e. α-cut is closed for all $\alpha \in [0,\ 1]$.Support of *A* is bounded. If left hand curve and right hand curve are straight lines then the fuzzy number is called triangular fuzzy number.

Definition 2. [1] For arbitrary fuzzy numbers A and B with α-cut $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ and $B_\alpha = [B_L(\alpha), B_U(\alpha)]$ respectively, the *distance* between *A* and *B* is defined by

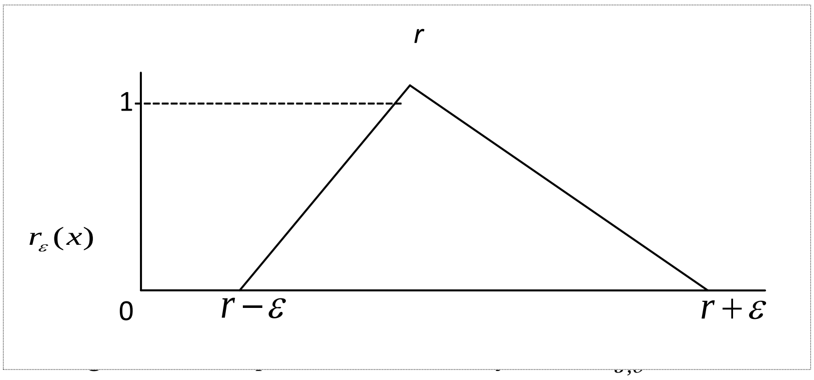
$$d(A,B) = \sqrt{\int_0^1 (|A_L(\alpha) - B_L(\alpha)|^2 + |A_U(\alpha) - B_U(\alpha)|^2) \mathrm{d}\alpha} .$$

Definition 3. [1] If *r* is a real number then ε fuzzy number r_ε, also called as symmetric epsilon fuzzy number is the triangular fuzzy number for some $\varepsilon \in \mathbb{R}$, (ε > 0) is a fuzzy set $r_\varepsilon : \mathbb{R} \to [0,1]$ defined by

$$r_\varepsilon(x) = \begin{cases} \dfrac{x-(r-\varepsilon)}{\varepsilon}, & \text{if } r-\varepsilon < x \le r, \\ \dfrac{x-(r+\varepsilon)}{-\varepsilon}, & \text{if } r < x \le r+\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

A triangular fuzzy number $A = (l, m, n)$ in above notation is denoted by $A = m_{m-l,n-m}$. Also, $r_\varepsilon = (r-\varepsilon, r, r+\varepsilon)$.

 ISSN: 2321-8169
934 - 939



The *support* of ε fuzzy number r_ε is $\left(r-\varepsilon, r+\varepsilon\right), r \in \mathbb{R}, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. The α-*cut* of r_ε is denoted by

.

$$(r_\varepsilon)_\alpha = [r - \varepsilon(1-\alpha), r + \varepsilon(1-\alpha)], \alpha \in (0.1]$$

The above notation for triangular fuzzy number is simple and may be considered as a family of functions of three parameters where left spread is identical with right spread.

Definition 4. [2] Let $r_{\varepsilon_1,\delta_1}$ and $s_{\varepsilon_2,\delta_2}$ be any two epsilon-delta fuzzy numbers where $r \le s$. If $r-\varepsilon_1 \le s-\varepsilon_2$ and $r+\delta_1 \le s+\delta_2$ then we define

$$(r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1\vee\varepsilon_2,\delta_1\wedge\delta_2} \text{ and. } r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2} = s_{\varepsilon_1\wedge\varepsilon_2,\,\delta_1\vee\delta_2}$$

If $s-\varepsilon_2 < r-\varepsilon_1$ and $r+\delta_1 \le s+\delta_2$ then we define

$$(r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_2-(s-r),\delta_1} \text{ and } (r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2}) = s_{\varepsilon_1+(s-r),\delta_2} \;.$$

If $r-\varepsilon_1 \le s-\varepsilon_2$ and $s+\delta_2 < r+\delta_1$ then

$$(r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_2+(s-r)} \text{ and } (r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2}) = s_{\varepsilon_2,\delta_1-(s-r)} \;.$$

If $s-\varepsilon_2 < r-\varepsilon_1$ and $s+\delta_2 < r+\delta_1$ then

$$r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2} = r_{\varepsilon_2-(s-r),\,\delta_2+(s-r)} \text{ and } (r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2}) = s_{\varepsilon_1+(s-r),\delta_1-(s-r)} \;.$$

The above evaluations are triangular approximations of max and min operations on fuzzy numbers. By setting $\varepsilon = \delta$ we get the results for the triangular epsilon fuzzy numbers. Many researchers have used Dubois and Prade approximation of max and min operations given below

$$r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\chi_2} = (r \wedge s)_{\varepsilon_1\vee\varepsilon_2,\,\delta_1\vee\delta_2} \text{ And } r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\chi_2} = (r \vee s)_{\varepsilon_1\wedge\varepsilon_2,\,\delta_1\vee\delta_2} \;.$$

Addition or subtraction of the term $|r-s|$ may be considered as a correcting factor and thereby gives better approximation.

ISSN: 2321-8169
934 - 939

3. Fuzzy two person zero sum game

The sets of the possible feasible strategies of player-I are two fuzzy sets *A* and *B* on S_1 and S_2 respectively. The two payoff functions P1 (for player-I) and P2 (for player-II) from $S_1 \times S_2 \to [0,1]$ are fuzzy.

Epsilon-delta Payoff matrix

Definition 5. A quantitative measure of satisfaction of a person expressed in terms of epsilon-delta fuzzy numbers is called epsilon-delta payoff

Suppose player *A* has *m* activities and player *B* has *n* activities. Then a fuzzy Payoff matrix can be formed by adopting following rules.

The player *A's* fuzzy payoff matrix

Player B

Player A

Strategies	1	2	…	j	…	n
1	$r^{11}_{\varepsilon_{11},\delta_{11}}$	$r^{12}_{\varepsilon_{12},\delta_{12}}$	….	$r^{1j}_{\varepsilon_{1j},\delta_{1j}}$	…..	$r^{1n}_{\varepsilon_{1n},\delta_{1n}}$
2	$r^{21}_{\varepsilon_{21},\delta_{21}}$	$r^{22}_{\varepsilon_{22},\delta_{22}}$	…..	$r^{2j}_{\varepsilon_{2j},\delta_{2j}}$	……	$r^{2n}_{\varepsilon_{2n},\delta_{2n}}$
: :	: :	: :	…..	: :	……	: :
i	$r^{i1}_{\varepsilon_{i1},\delta_{i1}}$	$r^{i2}_{\varepsilon_{i2},\delta_{i2}}$	……	$r^{ij}_{\varepsilon_{ij},\delta_{ij}}$	…….	$r^{in}_{\varepsilon_{in},\delta n}$
: :	: :	: :	…..	: :	……	: :
m	$r^{m1}_{\varepsilon_{m1},\delta_{m1}}$	$r^{m2}_{\varepsilon_{m2},\delta_{m2}}$	……	$r^{mj}_{\varepsilon_{mj},\delta_{mj}}$	…….	$r^{mn}_{\varepsilon_{mn},\delta_{mn}}$

We denote above fuzzy payoff matrix as follows

$$A_{E,D} = \begin{bmatrix} r^{11}_{\varepsilon_{11},\delta_{11}} & r^{12}_{\varepsilon_{12},\delta_{12}} & \cdots & r^{1n}_{\varepsilon_{1n},\delta_{1n}} \\ r^{21}_{\varepsilon_{21},\delta_{21}} & r^{22}_{\varepsilon_{22},\delta_{22}} & \cdots & r^{2n}_{\varepsilon_{2n},\delta_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ r^{m1}_{\varepsilon_{m1},\delta_{m1}} & r^{m2}_{\varepsilon_{m2},\delta_{m2}} & \cdots & r^{mn}_{\varepsilon_{mn},\delta_{mn}} \end{bmatrix}_{m\times n}, \quad \text{where } [A] = \begin{bmatrix} r^{11} & r^{12} & \cdots & r^{1n} \\ r^{21} & r^{22} & \cdots & r^{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r^{m1} & r^{m2} & \cdots & r^{mn} \end{bmatrix}_{m\times n},$$

$$E = [\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1n} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{m1} & \varepsilon_{m2} & \cdots & \varepsilon_{mn} \end{bmatrix}_{m\times n} \text{ and } D = [\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{m1} & \delta_{m2} & \cdots & \delta_{mn} \end{bmatrix}_{m\times n}$$

We note that for $\varepsilon = \delta$ we get respective matrices having triangular symmetric epsilon fuzzy numbers as their entries viz. the player *A's* fuzzy payoff matrix, $A_{E,D}$, E and D,

Fuzzy Minimax-Maximin Criterion

 ISSN: 2321-8169
934 - 939

Theorem 6. *Let, P be the payoff matrix for a two person zero sum game where* $P: X \to [0,1]$.

If $\vee_i \left(\wedge_j r_{\varepsilon_{ij}}^{ij} \right) = \underline{r}$ *and* $\wedge_j \left(\vee_i r_{\varepsilon_{ij}}^{ij} \right) = \overline{r}$ *then* $\overline{r} \geq \underline{r}$

i.e. $\wedge_j \left(\vee_i r_{\varepsilon_{ij}}^{ij} \right) \geq \vee_i \left(\wedge_j r_{\varepsilon_{ij}}^{ij} \right)$.

Proof. We have

$$\vee_i r_{\varepsilon_{ij}}^{ij} \geq r_{\varepsilon_{ij}}^{ij} \text{ And } \wedge_j r_{\varepsilon_{ij}}^{ij} \leq r_{\varepsilon_{ij}}^{ij}, \forall i, j$$

Let maximum be obtained at $i = i^*$ and minimum be obtained at $j = j^*$. Therefore,

$$r_{\varepsilon_{i^*j}}^{i^*j} \geq r_{\varepsilon_{ij}}^{ij} \geq r_{\varepsilon_{ij^*}}^{ij^*}, \forall i, j$$

$$\wedge_j \left(r_{\varepsilon_{i^*j}}^{i^*j} \right) \geq r_{\varepsilon_{ij}}^{ij} \geq \vee_i \left(r_{\varepsilon_{ij^*}}^{ij^*} \right), \forall i, j$$

$$\wedge_j \left(\vee_i r_{\varepsilon_{ij}}^{ij} \right) \geq \vee_i \left(\wedge_j r_{\varepsilon_{ij}}^{ij} \right)$$

Fuzzy Saddle Point

Definition 7. If

$$\wedge_j \left(\vee_i r_{\varepsilon_{ij}}^{ij} \right) = \vee_i \left(\wedge_j r_{\varepsilon_{ij}}^{ij} \right)$$

i.e. If $\overline{r} = \underline{r} = r_{\varepsilon_{ij}}^{ij}$, Then the fuzzy game has a saddle point at the cell (i, j).

4. Fuzzy optimal strategies

If fuzzy payoff $r_{\varepsilon_{ij}}^{ij}$ is a saddle point the players have fuzzy optimal strategies in pure strategies: Player I have i^{th} and player II have j^{th} fuzzy optimal strategies respectively.

Value of Fuzzy Game

The fuzzy payoff $r_{\varepsilon_{ij}}^{ij}$ at saddle point (i, j) is called value of fuzzy game.

Fair Fuzzy Game

A fuzzy game is said to be fair game if saddle point $r_{\varepsilon_{ij}}^{ij} = 0_{\varepsilon}$

Example 8. Consider two-person zero-sum game which represents fuzzy payoff to the player A. Find the optimal strategy if any

Player B

	I	II	III	Row Minimum
I	$-3_{9,9}$	$-2_{7,7}$	$6_{5,5}$	$-3_{9,5}$

Player A	II	$2_{7,7}$	$0_{4,4}$	$2_{7,7}$	$0_{7,4}$
	III	$5_{13,13}$	$-2_{7,7}$	$-4_{9,9}$	$-4_{11,7}$
Column Maximum		$5_{10,13}$	$0_{4,5}$	$6_{5,9}$	

Maximin Value $(\underline{r}) = 0_{7,5}$

Minimax Value $(\overline{r}) = 0_{5,5}$

$$\wedge_j \left(\vee_i r^{ij}_{\varepsilon_{ij},\delta_{ij}} \right) = 0_{5,5} \geq \vee_i \left(\wedge_j r^{ij}_{\varepsilon_{ij},\delta_{ij}} \right) = 0_{7,5}$$

i.e. $\overline{r} = \underline{r}$.

Thus, fuzzy optimal strategies of player-I and player-II is 2 and fuzzy game have saddle point $0_{5,5}$ or $0_{7,5}$ (since, $d(r_{\varepsilon,\delta}, 0_{5,5}) = d(r_{\varepsilon,\delta}, 0_{7,5})$.

5. Conclusion

In the present paper we considered a fuzzy two-person zero-sum game with triangular epsilon fuzzy payoffs having cell entries triangular epsilon fuzzy numbers. We defined the ordering of $a_{\varepsilon_1,\delta_1}$ and $b_{\varepsilon_2,\delta_2}$. The optimal strategies are obtained.

References

[1] Bapat, M. S., Yadav, S. N., Kamble, P. N., Triangular approximations of fuzzy numbers, International Journal of Statistika and Mathematatika, ISSN: 2277-2790 E-ISSN:2249-8605, Volume 7, Issue 3, (2013) 63-66.

[2] Bapat, M. S., Yadav, S. N., MAX-MIN Operations on Fuzzy Number,National Conference on Differntial Equation-2015,Shivaji University Kolhapur,29-30 January 2015

[3] Dubois, D., Prade, H., Fuzzy Sets and Systems: Theory and Applications, Academic Press, Boston, 1980.

[4] Nurmi, H. (1976).on fuzzy games, Eur.Meet.Cybern.Syst.Res.,3rd ,Vienna

[5] Orlovsky,S.A. (1977). On programming with fuzzy constraint sets.Kybernetes 6, pp 197-201

(Reference from II.4.)

[6] Ragade,R.K. (1976). Fuzzy games in the analysis of options.J.Cybern 6, pp213-221

[7] Sharma S D, Operation Research: Theory Methods and Applications, Kedar Nath Ram Nath Publication, 2013, ISBN-13 9789380803388.

[8] .Zadeh L. A., Fuzzy sets, Information and Control 8 (1965)338-353