Laplace Transformsand Its Applications

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Abstract: Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach just like the applications of transfer functions to solve ordinary differential equations. This paper will discuss the applications of Laplace transforms in the area of physics followed by the application to electric circuit analysis. A more complex application on Load frequency control in the area of power systems engineering is also discussed.

KeyWords:Laplace functions, Dirac delta functions, Inverse Laplace, linearity.

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I. INTRODUCTION:

The concept of Laplace transform are applied in area of science and technology such as electric analysis, communication engineering, control engineering, linear system analysis, statistics, optics, quantum physics etc. In solving problems relating to these fields Laplace transforms plays an important role and solves the problems in a systematic way. This paper provides the reader to know about the fundamentals of Laplace transform and some of the very basic applications to control systems.

Laplace Transforms:

Laplace transform is an operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

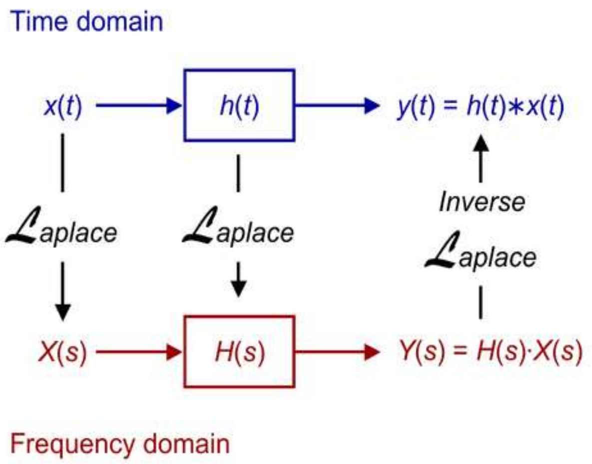
1) The given "hard" problem is transformed into a "simple" equation.
2) This simple equation is solved by purely algebraic manipulations.
3) The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem.

The Laplace transform is defined in the following way. Let f (t) is defined for $t \geq 0$. Then the Laplace transform of f, which is denoted by L[f(t)] or by F(s), is defined by the following equation

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

The Laplace transform can be interpreted as a transformation from the time domain where inputs and outputs are functions of time to the frequency domain where inputs and outputs are functions of complex angular frequency.



In order for any function of time f (t) to be Laplace transformable, it must satisfy the following Dirichlet conditions:

1) f(t) must be piecewise continuous which means that it must be single valued but can have a finite number of finite isolated discontinuities for t > 0.
2) f(t) must be exponential order which means that f(t) must remain less than $Se^{-a_0 t}$ as t approaches ∞ where S is a positive constant and $a_0$is a real Positive number.

If there is any function f(t) that satisfies the Dirichlet conditions then,

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

is called the Laplace transformation of f(t). Here, s can be either a real variable or acomplex quantity. The integral$\int f(t)e^{-st}dt$ converges if $\int|f(t)e^{-st}dt|<\infty$,$s = \sigma + j\omega$.

Properties and Theorems of Laplace Transforms:

The Laplace transforms of different functions is very important to solve mathematical problems in easiest possible way.

1) Linearity:
The Laplace transform of the linear sum of two Laplace transformable functions f(t) and g(t) is given as

$$L\big(af(t)+bg(t)\big) = aL\big(f(t)\big) + bL\big(g(t)\big)$$

$$L\big(af(t)+bg(t)\big) = aF(s) + bG(s)$$

2) Differentiation:
If the function f(t) is piecewise continuous so that it has a continuous derivative $f^{n-1}(t)$ of order n-1 and a continuous derivative $f^{n}(t)$ in every finite interval $0 \le t \le T$, then let, f(t) and all its derivatives through $f^{n-1}(t)$ be of exponential Order e^{ct} as $t \to \infty$.
Then, the transform of $f^{n}(t)$ exists when Re(s) > c and has the following form:
$L[f^{n}(t)] = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$

3) First Shifting Property:
If $L\{f(t)\}=F(s)$, when s>a then,
$L[e^{at}f(t)] = F[s-a]$
In words, the substitution s−a for s in the transform corresponds to the multiplication of the original function by e^{at}.

4) Second Shifting Property:
Suppose that f(t) has the transform F(s). Then the shifted function

$$F(t) = f(t-a).u(t-a) = \begin{cases} 0 & \text{if } t > a \\ f(t-a) & \text{if } t < a \end{cases}$$

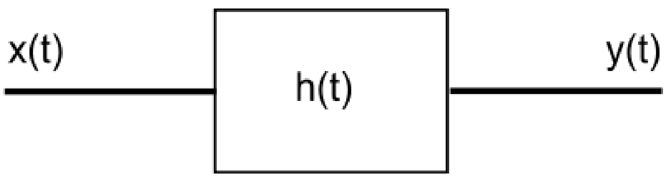
has the transform $e^{-as}F(s)$
$L[F(t)] = L[f(t-a).u(t-a)] = e^{-as}F(s)$

5) Laplace Transform of Integration:
If L[f(t)]=F[s] then, $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F[s]$

6) Scaling Property:
If L[f(t)]=F[s] then, $L[f(at)] = \frac{1}{a}F\left[\frac{s}{a}\right]$

7) Convolution Theorem:
If L[f(t)]=F[s] and If L[g(t)]=G[s] then Laplace of convolution f*g is

$$L[f*g](t) = \int_0^t f(u)g(t-u)du = F(s).G(s)$$



- *x(t)* = **system input**
- *h(t)* = **impulse response**
- *y(t)* = **system output**

8) Initial Value Theorem:

If f(t) and F(s) is Laplace transform pairs i.e. If L[f(t)]=F[s] then Initial value theorem is given by

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = f(0^+)$$

9) Final Value Theorem:
If f(t) and f '(t)both are Laplace Transformable and sF(s) has no pole in the R.H.P. (Right half Plane) then,
$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$

Formulae of Laplace Transforms:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{1}{(n-1)!}t^{n-1}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

Applications of Laplace Transforms in different fields:
1)Application in Physics:
An application of Laplace transform in physics is the harmonic vibration of a beam which is supported at its two ends .Let us consider a beam of length l and uniform cross section parallel to the y z plane so that the normal deflection w (x,t) is measured downward if the axis of the beam is towards x axis. The basic equation defining this phenomenon is as given below:

$$EI\frac{d^4w}{dx^4} - m\omega^2 w = 0\ldots(1)$$

Where E is Young's modulus of elasticity; I is the moment of inertia of the cross section with respect to the y axis; m is the mass per unit length; and ω is the angular frequency.
Let for convenience assume $\propto = \frac{m\omega^2 w}{EI}$ put in equation (1)

$$\frac{d^4\omega}{dx^4} - \propto^4 \omega = 0 \quad \ldots\ldots.(2)$$

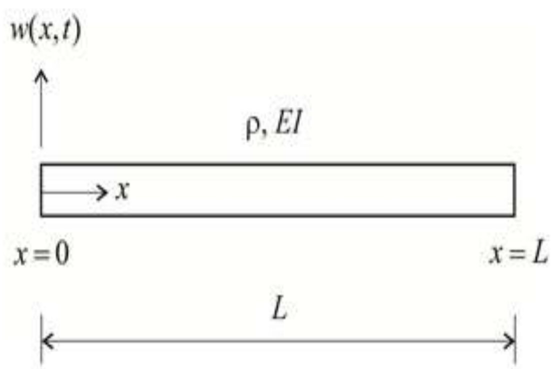


Figure: 01

Apply Laplace Transform to eq(2),
$s^4 f(s) - s^3 F(+0) - s^2 F'(+0) - sF''(+0) - sF'''(+0) - \propto^4 f(s) = 0$

The boundary conditions for this problem are:
$F(+0)=0, F(+l) = 0, F''(+0) = 0, F''(+l) = 0$

Hence, we obtain, the above eq. as

$$f(s) = \frac{s^2 F'(0+) + F'''(0+)}{s^4 - \propto^4}$$

Applying the inverse Laplace transform to above eq.

$$\omega = \frac{F'(+0)}{2\propto} \sinh \propto x + \sin \propto x + \frac{F'''(+0)}{2\propto^3} \sinh \propto x - \sin \propto x$$

$\omega = A_1 \sinh \propto x + A_2 \sin \propto x$

At the other end of beam where $x = l$,
$A_1 \sinh \propto l + A_2 \sin \propto l = 0,$
$\& A_1 \sinh \propto l - A_2 \sin \propto l = 0,$

These equations are satisfied when $A_1 = A_2 = 0$
i.e. $\sinh \propto l = \sin \propto l = 0$
i.e. $\propto l = n\pi$ for integral valuesof n.

Hence A_1=0 and A_2 is undetermined and the resulting vibrations are:
$\omega_n = \sin \frac{n\pi x}{l}$ and the frequencies are
$\omega_n = \frac{\pi^2 n^2}{l^2}\sqrt{\frac{EI}{M}}$, Here, if n = 1, it represents the fundamental vibration
and if n = 2 the first harmonic and so on.
Thus using Laplace Transform one can find harmonic vibration of a beam.

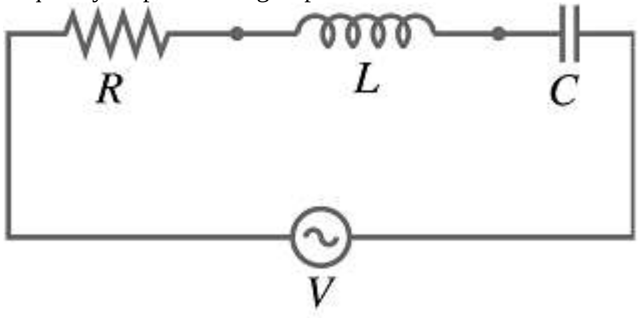
2)Application in Electrical Circuit:

We examine a number of basic circuits, solve the problem analytically usingthe Laplace transform, and then check our result empirically against an actualcircuit.

RLC Circuit:

The elegance of using the Laplace transform in circuit analysis lies in the automatic inclusion of the initial conditions in the transformation process, thus providing a complete (transient and steady state) solution.

Consider a series RLC circuit. We can determine its frequency response using Laplace Transform.



$$v_i(t) = R_i(t) + L_i'(t) + \frac{1}{C} q(t), \quad q(0) = 0, \quad i(0) = 0$$

where q(t) is the charge on the capacitor. To determineq(t) in termsofi(t). Recall that the charge on a capacitor and the current \through" thecapacitor is related by $q'(t) = i(t)$. By the fundamental theorem of calculus:

$$q(t) = q(0) + \int_0^t i(u)du$$

By our initial condition, we have q(0) = 0, so:

$$v_i(t) = R_i(t) + L_i'(t) + \frac{1}{C}\int_0^t i(u)du$$

We compare the input voltage to the voltage across theresistor:
$v_0(t) = R_i(t)$

We take the Laplace transform of vi(t):

$$v_i(s) = RI(s) + L[sI(s) - i(0)] + \frac{1}{sC} I(s)$$

$$v_i(s) = I(s)[R + Ls + \frac{1}{sC}]$$

Taking the Laplace transform of $v_0(t)$ is trivial:
$v_0(s) = RI(s)$

The transfer function is:

$$T(s) = \frac{v_0(s)}{v_i(s)} = \frac{RI(s)}{I(s)\left[R + Ls + \frac{1}{sC}\right]} = \frac{RCs}{LCs^2 + RCs + 1}$$

CONCLUSION:

The Laplace transform is an integral transformation of a function f(t) from the time domain into the complex frequency domain, F(s).The Laplace Transform is a powerful tool that is very useful in Electrical Engineering for solving differential equations analytically, especially in cases with discontinuous forcing terms or a periodic, non-sinusoidal forcing term. In addition, analysis of circuits in the s-domain can yield insights into the frequency response of the circuits.

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