Ricci Solitons in an (ϵ)-Kenmotsu Manifold Admitting Conharmonic Curvature Tensor

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Abstract: The object of the present paper is to study Ricci solitons in an (ϵ)-Kenmotsu manifold. In this paper, some curvature conditions of conharmonic curvature tensor and pseudo-projective curvaturetensor have been studied. Under these conditions taking ξ as space-like or time-like vector field, it is shown that Ricci solitons are expanding, steady or shrinking according as λ is positive, zero or negative respectively.

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1. Introduction:

In 1982,Hamilton [11] introduced the notion of Ricci flow to find a canonical metric on smooth manifolds. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds,especially for those manifoldswith positive curvature.Perelman [15]used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow.A solution to the Ricci flow is called Ricci solution if it moves onlyby a one parameter group of diffeomorphism and scaling. A Ricci soliton (g,V,λ) on a Riemannian manifold (M,g) is a generalization of an Einstein metric such that

$$\mathcal{L}_V g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \quad (1.1)$$

where S is a Ricci tensor, $\mathcal{L}_V$ is Lie-derivative operator along the vector field V on M andλ is a real number. The Ricci solitonis said to be shrinking, steady and expanding according as λ is negative,zero or positiverespectively. During the last two decades, the geometry of Ricci solitons has been the focus of attention ofmany mathematicians.In particular,it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904.In 2008, Sharma studied the Ricci solitons in contact geometry [18].Thereafter Ricci solitons incontact metric manifolds have been studied by various authors such as Bagewadi et. Al ([1],[2]),Bejan and Crasmareanu [3],Blaga[5],Chandra et. Al[6]Chen and Deshmukh[7],Deshmukh et.al[10],He and Zhu[13],Nagaraja and Premalatta[14]andmany others.

On the other hand, the study of manifolds with indefinite metrics is of interest from the stand point of physics and relativity. Manifolds with indefinite metrics have been studied byseveral authors. In 1993,Bejancu and Duggal [4] introduced the concept of (ϵ)-Sasakianmanifolds and Xufeng and Xiaoli [20] established that these manifolds are real hyper surfacesof indefinite Kahlerian manifolds. De and Sarkar[8]introduced (ϵ)-Kenmotsu manifolds and studied some curvature conditions on it. Singh,Pandey,Pandey and Tiwari [19] established the relation between semi-symmetric metric connection and Riemannian connection on (ϵ)-Kenmotsu manifolds and have studied several curvature conditions.

Motivated by these studies, we study Ricci solitons in (ϵ)-Kenmotsu manifolds.In this paper, we have studied Ricci solitons in (ϵ)-Kenmotsu manifolds satisfying $R(\xi,X).H=0$,$S(\xi,X).H=0$, $\bar{P}(\xi,X).H=0$ and $H(\xi,X).\bar{P}=0$,where H is a conharmonic curvature tensor,$\bar{P}$is a pseudo-projectivecurvature tensor.

2. Preliminaries

An n-dimensional smooth manifold (M^n, g) is called an (ϵ)-almost contact metric manifold

if

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\epsilon g(X, \xi) = \eta(X) \tag{2.3}$$

$$\epsilon = g(\xi, \xi) \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.5}$$

where ϵ is 1 or -1 according as ξ is space-like or time-like vector field and rank ϕ is n-1. It is important to mention that in the above definition ξ is never a light-like vector field.

If

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.6}$$

for every $X, Y \in TM^n$, then we say that M^n is an (ϵ)-contact metric manifold.

Also,

$$\phi\xi = 0 \text{ and } \eta o \phi = 0. \tag{2.7}$$

If an (ϵ)- contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.8}$$

where ∇ denotes the Riemannian connection of g, then M^n is called an $(\epsilon) -$ Kenmotsu

manifold [8]. An (ϵ)-almost contact metric manifold is an (ϵ)-Kenmotsu manifold

if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.9}$$

In an $(\epsilon) -$ Kenmotsu manifold, the following relations hold [8]

$$(\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.12}$$

$$R(X, Y)\phi Z = \phi R(X, Y)Z + \epsilon\{g(Y, Z)\phi X - g(X, Z)\phi Y + g(X, \phi Z)Y - g(Y, \phi Z)X\}, \tag{2.13}$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.14}$$

Let (g, V, λ) be a Ricci solitons in an (ϵ)-Kenmotsu manifold. From equation(2.9),we have

$$(L_\xi g)(X, Y) = -2[\epsilon g(X, Y) - \eta(X)\eta(Y)]. \tag{2.15}$$

In view of equations (1.1) and (2.15),we have

$$S(X,Y) = (\epsilon - \lambda)g(X,Y) - \eta(X)\eta(Y). \qquad (2.16)$$

The above equation yields that

$$QX = (\epsilon - \lambda)X - \epsilon\eta(X)\xi, \qquad (2.17)$$

$$S(X,\xi) = -\lambda\, g(X,\xi), \qquad (2.18)$$

$$r = n\,(\epsilon - \lambda) - \epsilon\,. \qquad (2.19)$$

The conharmoniccurvature tensor Hof type (1,3) on a Riemannian manifold (M^n, g) of dimensionnis defined by [9]

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \qquad (2.20)$$

Taking inner product of above equation with ξ and using equations (2.15),(2.16) and (2.17), the conharmonic curvature tensor on (ϵ)-Kenmotsu manifold takes the form

$$\eta(H(X,Y)Z) = [\epsilon + \frac{(\epsilon-2\lambda)}{(n-2)}][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \qquad (2.21)$$

Putting X=ξ in equation (2.20)and using equations (2.2),(2.3),(2.12),(2.16),(2.17) and (2.18),

we obtain

$$H(\xi,Y)Z = [1 - \frac{2\lambda}{n-2} + \frac{1}{n-2}][\eta(Z)Y - \epsilon g(Y,Z)\xi]. \qquad (2.22)$$

By putting Y=ξ in equation (2.20) and using equations (2.2), (2.3), (2.12)(2.17), (2.18)

and (2.19), we get

$$H(X,\xi)Z = [1 + \frac{\lambda\epsilon}{n-1} + \frac{1}{2(n-1)}][\epsilon\, g(X,Z)\xi - \eta(Z)X] + \lambda[g(X,Z)Y - \eta(Z)X]. \qquad (2.23)$$

Again by putting Z=ξ in equation (2.20) and using equations (2.2), (2.3), (2.11), (2.17) and

(2.18), we get

$$H(X,Y)\xi = [1 + \frac{1}{n-2} - \frac{2\lambda\epsilon}{n-2}][\eta(X)Y - \eta(Y)X]. \qquad (2.24)$$

Pseudo projective curvature tensor $\overline{P}$is defined by [16]

$$\overline{P}(X,Y)Z = a\,R(X,Y)Z + b[\,S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y]. \qquad (2.25)$$

Putting X=ξ in above equation and using equations (2.12),(2.16)and (2.18),we get

$$\overline{P}(\xi,Y)Z = [a + \lambda\epsilon b + \frac{r\epsilon}{n}\left(\frac{a}{n-1} + b\right)][\eta(Z)Y - \epsilon g(Y,Z)\xi] + b[\epsilon g(Y,Z)\xi - \eta(Y)\eta(Z)\xi]. \qquad (2.26)$$

Also by virtue of equation (2.25),we obtain

$$\overline{P}(X,Y)\xi = [a - b\lambda\epsilon + \frac{r\epsilon}{n}\left(\frac{a}{n-1} + b\right)][\eta(X)Y - \eta(Y)X]. \qquad (2.27)$$

Using equation (2.16) in equation (2.25),we get

$$\eta(\overline{P}(X,Y)Z) = [a\epsilon - (\epsilon - \lambda)b + \frac{r}{n}(\frac{a}{n-1} + b)][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \qquad (2.28)$$

Example: Consider 3-dimensional manifold $M=\{(x,y,z)\epsilon R^3; Z\neq 0\}$, where (x,y,z)

are the standard coordinates in R^3.

Let $\{e_1, e_{2,} e_3\}$ be linearly independent given by

$$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = -z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(e_1, e_3)$=$g(e_2, e_3)$=$g(e_1, e_2)$=0,

$g(e_1, e_1)$=$g(e_2, e_2)$=$g(e_3, e_3)$=ϵ,

where $\epsilon = \pm 1$, Let η be the $1-$form defined by $\eta(Z) = \epsilon g(Z, e_3)$ for any $Z \in TM^n$.

Let ϕ be the $(1,1)-$tensor field defined by

$$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0.$$

Then using the linearity property of ϕ and g we have

$$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any vector fields $U,W \in TM^n$.

Let ∇ be the Levi $-$ Civita connection with respect to metric g, we have

$$[e_{1,}e_2]=0, \quad [e_{1,}e_2]=\epsilon e_1, \quad [e_2, e_3] = \epsilon e_2.$$

The Riemannian connection ∇ of the metric g is given by Koszul's formula

$$2g(\nabla_X Y, Z)=X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),$$

and by virtue of it we have

$$\nabla_{e_1} e_3 = \epsilon e_1, \nabla_{e_2} e_3 = \epsilon e_2, \qquad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_3 = 0, \qquad \nabla_{e_2} e_2 = -\epsilon e_3, \qquad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_1 = -e_3, \qquad \nabla_{e_1} e_1 = 0, \qquad \nabla_{e_3} e_1 = 0,$$

for ξ=e_3. Hence the manifold under consideration is an $(\epsilon)-$ Kenmotsu manifolds of three –dimension.

3. Ricci solitons in an $(\epsilon)-$ Kenmotsumanifoldsatisfying R(ξ, X).H =0.

Let (R(ξ, X).H)(Y,Z)U=0, then we have

$$R(\xi, X)\, H(Y,Z)U- H(R(\xi, X)Y,Z)U- H(Y,R(\xi, X)Z)U- H(Y,Z)R(\xi, X)U=0. \tag{3.1}$$

By virtue of equations (2.12)and (2.20), above equation reduces to

$$\eta(H(Y, Z)U)X - \epsilon g(X, H(Y,Z)U)\xi - \eta(Y)H(X,Z)U + \epsilon g(X,Y)\, H(\xi, Z)U - \eta(Z)\, H(Y,X)U$$
$$+ \epsilon g(X,Z)\, H(Y,\xi)U- \eta(U)H(Y,Z)X + \epsilon g(X,U)\, H(Y,Z)\xi = 0. \tag{3.2}$$

Taking the inner product of above equation with ξ and using equations (2.2) and (2.3),

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we get

$\epsilon g(X, H(Y,Z)U)=\epsilon\eta(X)\eta(H(Y,Z)U) - \eta(Y)\eta(H(X,Z)U)+\epsilon g(X,Y)\eta(H(\xi,Z)U)$

$-\epsilon\eta(Z)\eta(H(Y,X)U)+\epsilon g(X,Z)\eta(H(Y,\xi)U)-\eta(U)\eta(H(Y,Z)X)$

$$+g(X,U)\eta(H(Y,Z)\xi). \quad (3.3)$$

In view of equation (2.21), above equation takes the form

$\epsilon g(X, H(Y,Z)U)= K[\epsilon g(Y,U)\eta(X)\eta(Z) - \epsilon g(Z,U)\eta(X)\eta(Y) + g(Z,U)\eta(X)\eta(Y)$

$$-\epsilon g(X,Y)g(Z,U)--g(Y,U)\eta(X)\eta(Z) + \epsilon g(Y,U)g(X,Z)], \quad (3.4)$$

where $K=[\epsilon + \frac{(\epsilon-2\lambda)}{(n-2)}]$.

By virtue of equation (2.20),above equation gives

$\epsilon' R(Y,Z,U)X - \frac{\epsilon}{(n-2)}[\epsilon S(Z,U)g(X,Y) - \epsilon S(Y,U)g(X,Z) + \epsilon g(Z,U)S(X,Y)\ \epsilon g(Y,U)S(X,Z)]$

$$= K[\epsilon g(Y,U)\eta(X)\eta(Z) - \epsilon g(Z,U)\eta(X)\eta(Y) + g(Z,U)\eta(X)\eta(Y)$$

$$-\epsilon g(X,Y)g(Z,U)--g(Y,U)\eta(X)\eta(Z) + \epsilon g(Y,U)g(X,Z)]. \quad (3.5)$$

where $K=[\epsilon + \frac{(\epsilon-2\lambda)}{(n-2)}]$.

Putting $X=U=e_i$ and taking summation over $i, 1 \le i \le n$, we get

$$\epsilon S(Y,Z)=0, \quad (3.6)$$

which on using equation (2.16) and by putting $Y=Z=\xi$, gives

$\lambda=0$.

This shows that λ is steady.Thus we can state as follows-

Theorem (3.1): A Ricci soliton in an $(\epsilon)-$ Kenmotsu manifold with ξ as space $-$ like vector field or time $-$ like vector field satisfying $R(\xi,X).H=0$, is steady.

4. Ricci solitons in an $(\epsilon)-$ Kenmotsumanifoldsatisfying $S(\xi,X).H=0$

Let $S(\xi,X).H=0$, which gives

$(S(X,\xi).H)(Y,Z)U=((X\wedge_s \xi)H)(Y,Z)U$

$=(X\wedge_s \xi)(H(Y,Z)U + H((X\wedge_s \xi)(Y,Z)U)$

$$+H(Y,(X\wedge_s \xi)Z)U + H(Y,Z)(X\wedge_s \xi)U, \quad (4.1)$$

where the endomorphism $(X\wedge_S Y)$ is defined by

$$(X\wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \quad (4.2)$$

Now, from equations (4.1) and (4.2), we get

$(S(X,\xi).H)(Y,Z)U = S(\xi,H(Y,Z)U)X - S(X,H(Y,Z)U)\xi + S(\xi,Y)H(X,Z)U)$

$$-S(X,Y)\,H(\xi,Z)U + S(\xi,Z)\,H(Y,X)U - S(X,Z)\,H(Y,\xi)U$$

$$+S(\xi,U)\,H(Y,Z)X - S(X,U)\,H\,(Y,Z)\xi. \qquad (4.3)$$

Assuming $(S(X,\xi).H)(Y,Z)W = 0$, then above equation reduces to

$$S(\xi,H(Y,Z)U)X - S(X,H\,(Y,Z)U)\xi + S(\xi,Y)H(X,Z)U - S(X,Y)H(\xi,Z)U$$

$$+S(\xi,Z)H(Y,X)U - S(X,Z)H(Y,\xi)U + S(\xi,U)H(Y,Y)X$$

$$-S(X,U)H(Y,Z)\xi = 0. \qquad (4.4)$$

Taking inner product of above equation with ξ, we get

$$\epsilon\eta(X)\,S(\xi,H(Y,Z)U) - S(X,H(Y,Z)U) + S(\xi,Y)\eta(\,H(X,Z)U) - S(X,Y)\eta(\,H(\xi,Z)U)$$

$$+S(\xi,Z)\eta(\,H(Y,X)U) - S(X,Z)\eta(\,H(Y,\xi)U) + S(\xi,U)\eta(\,H(Y,Z)X)$$

$$-S(X,U)\eta(H(Y,Z)\xi) = 0. \qquad (4.5)$$

In view of equations (2.16)and (2.18),above equation reduces to

$$(\epsilon-\lambda)\,g(X,H(Y,Z)U) = (1-\lambda)\eta(X)\eta(H(Y,Z)U) - \epsilon\lambda\eta(Y)\eta(H(X,Z)U) + \eta(X)\eta(Y)\eta(H(\xi,Z)U)$$

$$-\epsilon\lambda\eta(Z)\eta(H(Y,X)U) + \eta(X)\eta(Z)\eta(H(Y,\xi)U) - \epsilon\lambda\eta(U)\eta(H(Y,Z)X)$$

$$+\eta(X)\eta(U)\eta(\,H(Y,Z)\xi) - (\epsilon-\lambda)\{g(X,Y)\eta(H(\xi,Z)U) + g(X,Z)\eta(H(Y,\xi)U)$$

$$+g(X,U)\eta(H(Y,Z)\xi) = 0, \qquad (4.6)$$

which on using equation (2.21)gives

$$(\epsilon-\lambda)\,g(X,H(Y,Z)U) = K[(2-\lambda-\lambda\epsilon)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}$$

$$-\{g(X,Y)\eta(Z)\eta(U) - g(Z,X)\eta(Y)\eta(U)\}$$

$$-(\epsilon-\lambda)\{g(X,Y)\eta(Z)\eta(U) - g(Z,X)\eta(Y)\eta(U)\}], \qquad (4.7)$$

where $K = [\epsilon + \frac{(\epsilon-2\lambda)}{(n-2)}]$.

Now by use of equation (2.20), above takes the form

$$(\epsilon-\lambda)\,g(X,R\,(Y,Z)U) - \frac{(\epsilon-\lambda)}{(n-2)}[S(Z,U)g(X,Y) - S(Y,U)g(X,Z)$$

$$+g(Z,U)S(X,Y) - g(Y,U)S(X,Z)]$$

$$= K[(2-\lambda-\lambda\epsilon)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}$$

$$-\{g(X,Y)\eta(Z)\eta(U) - g(Z,X)\eta(Y)\eta(U)\}$$

$$-(\epsilon-\lambda)\{g(Y,U)g(X,Z) - g(X,Y)g(Z,U)\}], \qquad (4.8)$$

Using equation (2.16) in above equation, we get

$$(\epsilon-\lambda)R(Y,Z,U)X) - \frac{(\epsilon-\lambda)}{(n-2)}[(\epsilon-\lambda)\{g(Z,U)g(X,Y) - g(Y,U)g(X,Z)$$

$$+g(Z,U)g(X,Y) - g(Y,U)g(X,Z)\} - g(X,Y)\eta(Z)\eta(U)$$

$$+g(X,Z)\eta(Y)\eta(U) - g(Z,U)\eta(X)\eta(Y) + g(Y,U)\eta(X)\eta(Z)]$$

$$=K[(2-\lambda-\lambda\epsilon)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}$$

$$-\{g(X,Y)\eta(Z)\eta(U) - g(Z,X)\eta(Y)\eta(U)\}$$

$$-(\epsilon-\lambda)\{g(Y,U)g(X,Z) - g(X,Y)g(Z,U)\}]. \quad (4.9)$$

Putting X=U=e_i and taking summation over i,$1 \leq i \leq n$, we get

$$(\epsilon-\lambda)\, S(Y,Z) = 0, \quad (4.10)$$

which on using equation (2.16),gives

$$(\epsilon-\lambda)[(\epsilon-\lambda)g(Y,Z) - \eta(Y)\eta(Z)] = 0. \quad (4.11)$$

Putting Y=Z=ξin above equation,we get

$$\text{either} = 0 \; or \; \lambda = \frac{1}{\epsilon}. \quad (4.12)$$

Now,if ξ is space − like vector field in an (ϵ) − Kenmotsu manifolds, then from above equation (4.12), we obtain

$$\text{either} \quad \lambda = 0 \text{ or } \quad \lambda > 0,$$

which shows that either λ is steady or expanding .

Thus we can state as follows-

Theorem (4.1): A Ricci Soliton in an (ϵ)-Kenmotsu manifold with ξ as space − like vector field satisfying condition $S(\xi, X).H = 0$, is either steady or expanding.

Again, if we assume vector field ξ as time − like vector field in an (ϵ) − Kenmotsu manifolds, then in view of equation (4.12), we obtain

$$\text{either } \lambda = 0 \text{ or } \lambda < 0,$$

which shows that eitherλ is steady or shrinking.

Thus we can state as follows-

Theorem (4.2):A Ricci soliton in an (ϵ)-Kenmotsu manifold with ξ as time − like vector field satisfying condition $S(\xi, X).H = 0$, is either steady or shrinking.

5. Ricci solitons in an (ϵ) − Kenmotsumanifoldsatisfying$\bar{P}(\xi, X)$.H=0.

Let$\bar{P}(\xi, X)$.H =0, where $\bar{P}$ is pseudo-projective curvature tensor.

Then we have

$(\bar{P}(\xi, X).H)(Y,Z)U=0$,

which gives

$\bar{P}(\xi, X)H(Y,Z)U - H(\bar{P}(\xi, X)Y,Z)U - H(Y, \bar{P}(\xi, X)Z)W - H(Y,Z)\bar{P}(\xi, X)U=0$. (5.1)

Using equation (2.26) in above equation, we get

$$K_1[\eta(H(Y,Z)U)X - \epsilon g(X,H(Y,Z)U)\xi - \eta(Y)H(X,Z)U + \epsilon g(X,Y)H(\xi,Z)U$$

$$-\eta(Z)H(Y,X)U + \epsilon g(X,Z)H(Y,\xi)U - \eta(U)H(Y,Z)X + \epsilon g(X,U)H(Y,Z)\xi]$$

$$+b[\epsilon g(X,H(Y,Z)U)\xi - \eta(X)\eta(H(Y,Z)U)\xi + \epsilon g(X,Y)H(\xi,Z)U$$

$$-\eta(X)\eta(Y)H(\xi,Z)U + \epsilon g(X,Z)H(Y,\xi)U - \eta(X)\eta(Z)H(Y,\xi)U$$

$$+\epsilon g(X,U)H(Y,Z)\xi - \eta(X)\eta(U)H(Y,Z)\xi] = 0, \quad (5.2)$$

where $K_1 = \left[a + \lambda\epsilon b + \frac{r\epsilon}{n}\left(\frac{a}{n-1} + b\right)\right]$.

Taking the inner product of above equation with ξ, we get

$$K_1[\epsilon\eta(X)\eta(H(Y,Z)U) - \epsilon g(X,H(Y,Z)U) - \eta(Y)\eta(H(X,Z)U) + \epsilon g(X,Y)\eta(H(\xi,Z)U)$$

$$-\eta(Z)\eta(H(Y,X)U) + \epsilon g(X,Z)\eta(H(Y,\xi)U) - \eta(U)\eta(H(Y,Z)X) + \epsilon g(X,U)\eta(H(Y,Z)\xi)]$$

$$+b[\epsilon g(X,H(Y,Z)U) - \eta(X)\eta(H(Y,Z)U)\xi + \epsilon g(X,Y)\eta(H(\xi,Z)U)$$

$$-\eta(X)\eta(Y)\eta(H(\xi,Z)U) + \ + \epsilon g(X,Z)\eta(H(Y,\xi)U) - \eta(X)\eta(Z)\eta(H(Y,\xi)U)$$

$$+\epsilon g(X,U)\eta(H(Y,Z)\xi) - \eta(X)\eta(U)\eta(H(Y,Z)\xi)] = 0. \quad (5.3)$$

In view of equation (2.21), above equation reduces to

$$\epsilon(K_1 - b)g(X,H(Y,Z)U) = K_1K_2[(\epsilon - 1)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}$$

$$+\epsilon\{g(X,Z)\eta(Y)\eta(U) - g(X,Y)g(Z,U)\}$$

$$+K_1b(\epsilon - 1)[g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(U)\eta(Z)] = 0, \quad (5.4)$$

which on using equation (2.21), gives

$$\epsilon(K_1 - b)R(Y,Z,U)X - \frac{(K_1-b)\epsilon}{n-2}[S(Z,U)S(X,Y) - S(Y,U)g(X,Z) + g(Z,U)S(X,Y)$$

$$-g(Y,U)S(X,Z)] = K_1K_2[(\epsilon - 1)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}$$

$$+\epsilon\{g(X,Z)\eta(Y)\eta(U) - g(X,Y)g(Z,U)\}$$

$$+K_1b(\epsilon - 1)[g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(U)\eta(Z)] = 0. \quad (5.5)$$

Usingequation (2.16) in above equation, we get

$$(K_1 - b)\epsilon R(Y,Z,U)X - \frac{(K_1-b)\epsilon}{n-2}[2(\epsilon - \lambda)\{g(Z,U)g(X,Y) - g(Y,U)g(X,Z)$$

$$+g(Z,U)g(X,Y) - g(Y,U)g(X,Z)\} - g(X,Y)\eta(Z)\eta(U)$$

$$+g(X,Z)\eta(Y)\eta(U) - g(Z,U)\eta(X)\eta(Y) + g(Y,U)\eta(X)\eta(Z)]$$

$$= K_1K_2[(\epsilon - 1)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}$$

$$+\epsilon\{g(X,Z)\eta(Y)\eta(U) - g(X,Y)g(Z,U)\}$$

$$+K_1b(\epsilon - 1)[g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(U)\eta(Z)] = 0. \quad (5.6)$$

Putting X=U=e_i and taking summation over i,1 ≤ i ≤n, we get

$$\epsilon(K_1 - b)S(Y,Z) = 0. \tag{5.7}$$

Using equation (2.17) in above equation, we get

$$(K_1 - b)\epsilon[(\epsilon - \lambda)g(Y,Z) - \eta(Y)\eta(Z)] = 0, \tag{5.8}$$

Putting Y=Z=ξ above equation, we get

either $\lambda = 0$ or $(K_1 - b) = 0$,(5.9)

Now, supposeξ is space like vector fields in an (ϵ) − Kenmotsu manifolds, then from above equation (5.9), we obtain,

either$\lambda = 0$ or $\lambda < 0$,

which shows that λ is either steady or shrinking. Thus we can state as follows-

Theorem (5.1):A Ricci soliton in an (ϵ)-Kenmotsu manifold with ξ as space − like vector field satisfying condition $\bar{P}(\xi, X).H = 0$, is either steady or shrinking.

Again if we assume vector field ξ as time − like vector field in an (ϵ) − Kenmotsu manifolds, then in view of equation (5.9), we obtain

$$\lambda = 0 \text{ or} \lambda > 0,$$

which shows that λ is a either steady or expanding.

Thus we can state as follows-

Theorem(5.2): A Ricci soliton in an (ϵ) − Kenmotsumanifold admitting ξ as time like vector fieldsatisfying$\bar{P}(\xi, X).H = 0$, is either steady or expanding.

6. Ricci solitons in an (ϵ) − Kenmotsumanifold satisfyingH(ξ , X).$\bar{P}$=0.

The condition $H(\xi, X).\bar{P} = 0$, implies that

$(H(\xi,X).\overline{P)}(Y,Z)U = 0$,

which gives

$$H(\xi, X)\bar{P}(Y,Z)U - \bar{P}(H(\xi, X)Y,Z)U - \bar{P}(Y, H(\xi, X)Z)U - \bar{P}(Y,Z)H(\xi, X)U = 0, \tag{6.1}$$

Using equation (2.22)in above equation,we get

$$\eta(\bar{P}(Y,Z)U)X - \epsilon g(X, \bar{P}(Y,Z)U)\xi - \eta(Y)\bar{P}(X,Z)U + \epsilon g(X,Y)\bar{P}(\xi,Z)U$$

$$-\eta(Z)\bar{P}(Y,X)U + \epsilon g(X,Z)\bar{P}(Y,\xi)U - \eta(U)\bar{P}(Y,Z)X + \epsilon g(X,U)\bar{P}(Y,Z)\xi] = 0, \tag{6.2}$$

where $K=\left[1 - \frac{2\lambda}{n-2} + \frac{1}{n-2}\right]$.

Taking the inner product of above equation with ξ, we get

$$\epsilon\eta(X)\eta(\bar{P}(Y,Z)U) - \epsilon g(X, \bar{P}(Y,Z)U) - \eta(Y)\eta(\bar{P}(X,Z)U) + \epsilon g(X,Y)\eta(\bar{P}(\xi,Z)U)$$

$-\eta(Z)\eta(\bar{P}(Y,X)U)+\epsilon g(X,Z)\eta(\bar{P}(Y,\xi)U)-\eta(U)\eta(\bar{P}(Y,Z)X)$

$$+\epsilon g(X,U)\eta(\bar{P}(Y,Z)\xi)]=0. \quad (6.3)$$

In view of equation (2.28) above equation reduces to

$$\epsilon g(X,\bar{P}(Y,Z)U)=K[\epsilon g(X,Z)g(Y,U)-\epsilon g(X,Y)g(Z,U)], \quad (6.4)$$

which on using equation (2.25) gives

$$\epsilon\, a g(X,R(Y,Z)U)+\epsilon b[S(Z,U)g(X,Y)-S(Y,U)g(X,Z)]$$

$$-\frac{r\epsilon}{n}\left[\frac{a}{n-1}+b\right][g(Z,U)g(X,Y)-g(Y,U)g(X,Z)]$$

$$=K[g(Y,U)\eta(Z)-g(Z,U)\eta(Y)]. \quad (6.5)$$

Using equation (2.16) in above equation, we get

$$\epsilon\, a\, g(X,R(Y,Z)U)+\epsilon b[(\epsilon-\lambda)\{g(X,Y)g(Z,U)-g(Y,U)g(X,Z)\}$$

$$-g(X,Y)\eta(Z)\eta(U)+g(X,Z)\eta(Y)\eta(U)]$$

$$-\frac{r\epsilon}{n}\left[\frac{a}{n-1}+b\right][g(Z,U)g(X,Y)-g(Y,U)g(X,Z)]$$

$$=K[g(Y,U)\eta(Z)-g(Z,U)\eta(Y)]. \quad (6.6)$$

Putting $X=U=e_i$ and taking summation over $i, 1\le i\le n$, we get

$$\epsilon\, a\, S(Y,Z)=K[g(Y,e_i)\eta(Z)-g(Z,e_i)\eta(Y)], \quad (6.7)$$

which on using equation (2.16), gives

$$\epsilon a[(\epsilon-\lambda)g(Y,Z)-\eta(Y)\eta(Z)]=K[g(Y,e_i)\eta(Z)-g(Z,e_i)\eta(Y)]. \quad (6.8)$$

Putting $Y=Z=\xi$ in above equation, we get

$$\lambda=0,$$

which shows that λ is steady.

Thus we can state as follows-

Theorem (6.1): A Ricci soliton in an (ϵ)-Kenmotsu manifold with ξ as space − like vector field satisfying condition $H(\xi,X).\bar{P}=0$, is steady.

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