Notes on Interval-Valued Hesitant Fuzzy Soft Topological Space

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Abstract: In this paper we introduce the notion of interval valued hesitant fuzzy soft topological space. Also the concepts of interval valued hesitant fuzzy soft closure; interior and neighbourhood are introduced here and established some important results.

Keywords: Fuzzy soft sets; Interval-valued hesitant fuzzy sets; Interval-valued hesitant fuzzy soft sets; Interval-valued hesitant fuzzy soft topological space.

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1. Introduction

The concept of interval arithmetic was first suggested by Dwyer [10] in 1951. Chiao[9] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. A set is called an interval number if it consisting of a closed interval of real numbers \(x\) such that \(a \leq x \leq b\). A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by \(IR\). Any elements of \(IR\) is called closed interval and denoted by \(x\). That is \(\bar{x} = \{x \in \mathbb{R}: a \leq x \leq b\}\). An interval number \(\bar{x}\) is a closed subset of real numbers (see [9]). Let \(x_i\) and \(x\) be first and last points of \(\bar{x}\) interval number, respectively.

For \(\bar{x}_1, \bar{x}_2 \in IR\), we have \(\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{i1} = x_{2l}, x_{i1} = x_{2r}\). \(\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R}: x_{i1} + x_{2l} \leq x \leq x_{i1} + x_{2r}\}\), and if \(\alpha \geq 0\), then \(\alpha \bar{x} = \{x \in \mathbb{R}: \alpha x_{i1} \leq x \leq \alpha x_{i1}\}\) and if \(\alpha < 0\), then \(\alpha \bar{x} = \{x \in \mathbb{R}: \alpha x_{i1} \leq x \leq \alpha x_{i1}\} \times \min\{x_{i1}x_{2l}, x_{i1}x_{2r}, x_{i1}x_{2r}, x_{i1}x_{2r}\} \leq x \leq \max\{x_{i1}x_{2l}, x_{i1}x_{2r}, x_{i1}x_{2r}, x_{i1}x_{2r}\}\}.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets, introduced by L.A. Zadeh [31] in 1965. This theory brought a paradigmatic change in mathematics. But there arise difficulty that how to set the membership function in each particular case. The Hesitant fuzzy set, as one of the extensions of Zadeh [31] fuzzy set, allows the membership degree that an element to a set presented by several possible values, and it can express the hesitant information more comprehensively than other extensions of fuzzy sets. Torra and Narukawa [24] introduced the concept of hesitant fuzzy set. Xu and Xia [30] defined the concept of hesitant fuzzy element, which can be considered as the basic unit of a hesitant fuzzy set, and is a simple and effective tool used to express the decision makers’ hesitant preferences in the process of decision making. So many researchers has done lots of research work on aggregation, distance, similarity and correlation measures, clustering analysis, and decision making with hesitant fuzzy information. Babitha and John [3] defined another important soft set i.e. Hesitant fuzzy soft sets. They introduced basic operations such as intersection, union, compliment and De Morgan’s law was proved. Chen et al. [8] extended hesitant fuzzy sets into interval-valued hesitant fuzzy environment and introduced the concept of interval-valued hesitant fuzzy sets. Zhang et al. [32] introduced some operations such as complement, "AND", "OR", ring sum and ring product on interval-valued hesitant fuzzy soft sets.

There are many theories like theory of probability, theory of fuzzy sets, and theory of intuitionistic fuzzy sets, theory of rough sets etc. which can be considered as mathematical tools for dealing with uncertain data, obtained in various fields of engineering, physics, computer science, economics, social science, medical science, and of many other diverse fields. But all these theories have their own difficulties. The theory of intuitionistic fuzzy sets (see[1, 2]) is a more generalized concept than the theory of fuzzy sets, but this theory has the same difficulties. All the above mentioned theories are successful to some extent in dealing with
problems arising due to vagueness present in the real world. But there are also cases where these theories failed to give satisfactory results, possibly due to inadequacy of the parameterization tool in them. As a necessary supplement to the existing mathematical tools for handling uncertainty, Molodtsov [16] introduced the theory of soft sets as a new mathematical tool to deal with uncertainties while modelling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. Molodtsov et al. [17] successfully applied soft sets in directions such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, and theory of measurement. Maji et al [13] gave the first practical application of soft sets in decision-making problems. Maji et al [14] defined and studied several basic notions of the soft set theory. Also Cagman et al [6] studied several basic notions of the soft set theory. V. Torra[23, 24] and Verma and Sharma [25] discussed the relationship between hesitant fuzzy set and intuitionistic fuzzy set. Zhang et.al [32] introduced weighted interval-valued hesitant fuzzy soft sets and finally applied it indecision making problem. The notion of topological space is defined on crisp sets and hence it is affected by different generalizations of crisp sets like fuzzy sets and soft sets. In 1968, C. L. Chang [7] introduced fuzzy topological space and in 2011, subsequently Cagman et al. [6] and Shabir et al. [21] introduced fuzzy soft topological spaces and studied neighborhood. Mahanta et al. [12], Neog et al. [18] and Ray et. al [20] introudused fuzzy soft topological spaces in different direction.

In this paper, in section 3, first we give a counter example of equality of IVHFSSps proposed by Zhang et al. [32]. Secondly we point out that proposition 3.11 in a previous paper by Borah and Hazarika [4] true in general by counter example. Thirdly we introduce about notion of topological space.

2. Preliminaries and Definitions

In this section we recall some basic concepts and definitions regarding fuzzy soft sets, hesitant fuzzy set and hesitant fuzzy soft set.

Definition 2.1. [15] Let U be an initial universe and F be a set of parameters. Let \( P(U) \) denote the power set of U and A be a non-empty subset of F. Then \( F_A \) is called a fuzzy soft set over U, where \( F : A \rightarrow \tilde{P}(U) \) is a mapping from A into \( \tilde{P}(U) \).

Definition 2.2. [16] \( F_E \) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U.

In other words, the soft set is a parameterized family of subsets of the set U. Every set \( F(\varepsilon), \varepsilon \in E \), from this family may be considered as the set of \( \varepsilon \) – element of the soft set \( F_E \) of the set \( \varepsilon \) – approximate elements of the set soft.

Definition 2.3. [2, 28] Let intuitionistic fuzzy value \( IFV(X) \) denote the family of all IFVs defined on the universe X and let \( \alpha, \beta \in IFV(X) \) be given as:

\[
\alpha = (\mu_\alpha, \nu_\alpha), \beta = (\mu_\beta, \nu_\beta),
\]

\[
\begin{align*}
(i) \quad & \alpha \cap \beta = (\min(\mu_\alpha, \mu_\beta), \max(\nu_\alpha, \nu_\beta)) \\
(ii) \quad & \alpha \cup \beta = (\max(\mu_\alpha, \mu_\beta), \min(\nu_\alpha, \nu_\beta)) \\
(iii) \quad & \alpha * \beta = \left(\frac{\mu_\alpha + \mu_\beta}{2(\mu_{\beta_1}), \frac{\nu_\alpha + \nu_\beta}{2(\nu_{\beta_1})}\right)
\end{align*}
\]

Definition 2.4. [23] Given a fixed set X, then a hesitant fuzzy set (shortly HFS) in X is in terms of a function that when applied to X returns a subset of \([0, 1]\). We express the HFS by a mathematical symbol:

\[
F = \{h, \mu_F(x) : x \in X\}, \quad \mu_F(x) \text{ is a set of some values in } [0, 1],
\]

denoting the possible membership degrees of the element \( h \in X \) to the set \( F \). \( \mu_F(x) \) is called a hesitant fuzzy element (HFE) and H is the set of all HFEs.

Definition 2.5. [23] Let \( \mu_1, \mu_2 \in H \) and three operations are defined as follows:

\[
\begin{align*}
(i) \quad & \mu_1^C = \cup_{\gamma_1 \in \mu_1} (1 - \gamma_1) \\
(ii) \quad & \mu_1 \cup \mu_2 = \cup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \max\{\gamma_1, \gamma_2\} \\
(iii) \quad & \mu_1 \cap \mu_2 = \cap_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \min\{\gamma_1, \gamma_2\}
\end{align*}
\]

Definition 2.6. [8] Let X be a reference set, and \( D[0, 1] \) be the set of all closed subintervals of \([0, 1]\). An IVFHS on X is \( F = \{h_i, \mu_F(h_i) : h_i \in X, i = 1, 2, \ldots, n\}, \) where \( \mu_F(h_i) : X \rightarrow D[0, 1] \) denotes all possible interval-valued membership
degrees of the element \( h_i \in X \) to the set \( F \). For convenience, we call \( \mu_F(h_i) \) an interval-valued hesitant fuzzy element (IVHFE), which reads \( \mu_F(h_i) = \{ \gamma : \gamma \in \mu_F(h_i) \} \).

Here \( \gamma = [\gamma^L, \gamma^U] \) is an interval number. \( \gamma^L = \inf \gamma \) and \( \gamma^U = \sup \gamma \) represent the lower and upper limits of \( \gamma \) respectively. An IVHFE is the basic unit of an IVHFS and it can be considered as a special case of the IVHFS. The relationship between IVHFE and IVHFS is similar to that between interval-valued fuzzy number and interval-valued fuzzy set.

Example 2.7. Let \( U = \{ h_1, h_2 \} \) be a reference set and let \( \mu_F(h_i) = \{ [0.6, 0.8], [0.2, 0.7] \} \) \( \mu_F(h_2) = \{ [0.1, 0.4] \} \) be the IVHFEs of \( h_i(i = 1, 2) \) to a set \( F \), respectively. Then IVHFS \( F \) can be written as \( F = \{ < h_1, \{ [0.6, 0.8], [0.2, 0.7] \} >, < h_2, \{ [0.1, 0.4] \} > \} \).

Definition 2.8. [29] Let \( \tilde{a} = [a^L, a^U] \) and \( \tilde{b} = [b^L, b^U] \) be two interval numbers and \( \lambda \geq 0 \), then

(i) \( \tilde{a} + \tilde{b} = [a^L + b^L, a^U + b^U] \);

(ii) \( \lambda \tilde{a} = [\lambda a^L, \lambda a^U] \).

(iii) \( \lambda \tilde{a} = [\lambda a^L, \lambda a^U] \), especially \( \lambda \tilde{a} = 0 \) if \( \lambda = 0 \).

Definition 2.9. [29] Let \( \tilde{a} = [a^L, a^U] \) and \( \tilde{b} = [b^L, b^U] \) and let \( l_a = a^U - a^L \) and \( l_b = b^U - b^L \); then the degree of possibility of \( \tilde{a} \geq \tilde{b} \) is formulated by

\[
p(\tilde{a} \geq \tilde{b}) = \max \left\{ 1 - \frac{b^U - a^L}{l_a + l_b}, 0 \right\}
\]

Above equation is proposed in order to compare two interval numbers and to rank all the input arguments.

Definition 2.10. [8] For an IVHFE \( \tilde{\mu}, s(\tilde{\mu}) = \frac{1}{l_{\tilde{\mu}}} \sum_{\gamma \in \tilde{\mu}} \gamma \) is called the score function of \( \mu \) with \( l_{\tilde{\mu}} \) being the number of the interval values in \( \tilde{\mu} \), and \( s(\tilde{\mu}) \) is an interval value belonging to \( [0, 1] \). For two IVHFEs \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \), if \( s(\tilde{\mu}_1) \geq s(\tilde{\mu}_2) \) then \( \tilde{\mu}_1 \geq \tilde{\mu}_2 \). We can judge the magnitude of two IVHFEs using above equation.

Definition 2.11. [8] Let \( \mu, \mu_1, \) and \( \mu_2 \) be three IVHFEs, then

(i) \( \tilde{\mu}_1 \cup \tilde{\mu}_2 = [\max(\gamma_1^L, \gamma_2^L), \max(\gamma_1^U, \gamma_2^U)]; \gamma_1 \in \tilde{\mu}_1, \gamma_2 \in \tilde{\mu}_2 \);

(ii) \( \tilde{\mu}_1 \cap \tilde{\mu}_2 = [\min(\gamma_1^L, \gamma_2^L), \min(\gamma_1^U, \gamma_2^U)]; \gamma_1 \in \tilde{\mu}_1, \gamma_2 \in \tilde{\mu}_2 \);

(iii) \( \tilde{\mu}_1 \ominus \tilde{\mu}_2 = [(\gamma_1^L + \gamma_2^L - \gamma_1^U, \gamma_2^L, \gamma_1^U + \gamma_2^L - \gamma_1^U); \gamma_1 \in \tilde{\mu}_1, \gamma_2 \in \tilde{\mu}_2 \);

(iv) \( \tilde{\mu}_1 \oplus \tilde{\mu}_2 = [(\gamma_1^L + \gamma_2^L, \gamma_1^U, \gamma_2^L, \gamma_1^U); \gamma_1 \in \tilde{\mu}_1, \gamma_2 \in \tilde{\mu}_2 \);

(v) \( \tilde{\mu}_1 \odot \tilde{\mu}_2 = [(\gamma_1^L, \gamma_1^U, \gamma_2^L, \gamma_2^U)]; \gamma_1 \in \tilde{\mu}_1, \gamma_2 \in \tilde{\mu}_2 \);

Proposition 2.12. [8] For three IVHFEs \( \mu, \mu_1 \) and \( \mu_2 \), we have

(i) \( \tilde{\mu}_1 \cup \tilde{\mu}_2 = (\tilde{\mu}_1 \cap \tilde{\mu}_2)^C \);

(ii) \( \tilde{\mu}_1 \cap \tilde{\mu}_2 = (\tilde{\mu}_1 \cup \tilde{\mu}_2)^C \);

Definition 2.13. [26] Let \( U \) be an initial universe and \( E \) be a set of parameters. Let \( \tilde{F}(U) \) be the set of all hesitant fuzzy subsets of \( U \). Then \( F_E \) is called a hesitant fuzzy soft set (HFSS) over \( U \), where \( F : E \rightarrow \tilde{F}(U) \).

A HFSS is a parameterized family of hesitant fuzzy subsets of \( U \), that is \( \tilde{F}(U) \). For all \( \varepsilon \in E, F(\varepsilon) \) is referred to as the set of \( \varepsilon - \) approximate elements of the HFSS \( F_E \). It can be written as
\[ F(e) = \{ < h, \mu_{F(e)}(x) > : h \in U \} , \]

Since HFE can represent the situation, in which different membership function are considered possible (see [23]), \( \mu_{F(e)}(x) \) is a set of several possible values, which is the hesitant fuzzy membership degree. In particular, if \( F(e) \) has only one element, \( F(e) \) can be called a hesitant fuzzy soft number. For convenience, a hesitant fuzzy soft number (HFSN) is denoted by \( \{ < h, \mu_{F(e)}(x) > \} \).

Example 2.14. Suppose \( U = \{ h_1, h_2 \} \) be an initial universe and \( E = \{ e_1, e_2, e_3 \} \) be a set of parameters. Let \( A = \{ e_1, e_2 \} \).

Then the hesitant fuzzy soft set \( F_A \) is given as
\[ F_A = \{ F(e_1) = \{ < h_1, [0.6,0.8] >, < h_2, [0.8,0.4,0.9] > \} \]
\[ F(e_2) = \{ < h_1, [0.9,0.1,0.5] >, < h_2, [0.2] > \} \].

Definition 2.15. [32] Let \( (U,E) \) be a soft universe and \( A \subseteq E \). Then \( F_A \) is called an intervalvalued hesitant fuzzy soft set over \( U \), where \( F \) is a mapping given by \( F : A \rightarrow IVHF(U) \). An interval-valued hesitant fuzzy soft set is a parameterized family of interval-valued hesitant fuzzy subset of \( U \). That is to say, \( F(e) \) is an interval-valued hesitant fuzzy subset in \( U, \forall e \in A \). Following the standard notations, \( F(e) \) can be written as
\[ \tilde{F}(e) = \{ < h, \mu_{F(e)}(x) > : h \in U \} . \]

Example 2.16. Suppose \( U = \{ h_1, h_2 \} \) be an initial universe and \( E = \{ e_1, e_2, e_3 \} \) be a set of parameters. Let \( A = \{ e_1, e_2 \} \).

Then the interval valued hesitant fuzzy soft set \( F_A \) is given as
\[ F_A = \{ F(e_1) = \{ < h_1, [0.6,0.8] >, < h_2, [0.1,0.4] > \} \]
\[ e_2 = \{ < h_1, [0.2,0.6], [0.3,0.9] >, < h_2, [0.2,0.5], [0.2,0.8], [0.2,0.8] > \} \].

Definition 2.17. [32] Let \( U \) be an initial universe and let \( E \) be a set of parameters. Supposing that \( A, B \subseteq E, F_A \) and \( F_B \) are two interval-valued hesitant fuzzy soft sets, one says that \( F_A \) is an interval-valued hesitant fuzzy soft subset of \( G_B \) if and only if
(i) \( A \subseteq B \),
(ii) \( \gamma_1^{\sigma(k)} \geq \gamma_2^{\sigma(k)} \)

Where for all \( e \in A, \gamma_1^{\sigma(k)} \) and \( \gamma_2^{\sigma(k)} \) stand for the kth largest interval number in the IVHFEs \( \mu_{F(e)}(x) \) and \( \mu_{G(e)}(x) \) respectively.

In this case, we write \( F_A \subseteq G_A \).

Definition 2.18. [32] The complement of \( F_A \), denoted by \( F_A^C \), is defined by
\[ \tilde{F}_A^C(e) = \{ < h, \mu_{F_A^C(e)}(x) > : h \in U \} \]
where \( \mu_{F_A^C(e)} : A \rightarrow IVHF(U) \) is a mapping given by \( \mu_{F_A^C(e)} \forall e \in A \) such that \( \mu_{F_A^C(e)} \) is the complement of interval-valued hesitant fuzzy element \( \mu_{F(e)} \) on \( U \).

Definition 2.19. [32] An interval-valued hesitant fuzzy soft set is said to be an empty interval-valued hesitant fuzzy soft set, denoted by \( \tilde{\phi} \), if \( F : E \rightarrow IVHF(U) \) such that
\[ \tilde{F}(e) = \{ < h, \mu_{F(e)}(x) > : h \in U \} = \{ < h, [0,0] > : h \in U \} \forall e \in E \].

Definition 2.20. [32] An interval-valued hesitant fuzzy soft set is said to be a full interval-valued hesitant fuzzy soft set, denoted by \( \tilde{E} \), if \( F : E \rightarrow IVHF(U) \) such that
\[ \tilde{F}(e) = \{ < h, \mu_{F(e)}(x) > : h \in U \} = \{ < h, [1,1] > : h \in U \} \forall e \in E \].
Definition 2.21. [4] The union of two interval-valued hesitant fuzzy soft sets $F_A$ and $G_B$ over $(U, E)$, is the interval-valued hesitant fuzzy soft set $H_C$, where $C = A \cup B$ and, $\forall e^h C$,

$$\mu_{H(e)} = \begin{cases} 
\mu_{F(e)^h}, \text{if } e^h \in A - B; \\
\mu_{G(e)^h}, \text{if } e^h \in B - A; \\
\mu_{F(e)^h} \cup \mu_{G(e)^h}, \text{if } e^h \in A \cap B.
\end{cases}$$

We write $F_A \bigcup G_B = H_C$.

Definition 2.22. [4] The intersection of two interval-valued hesitant fuzzy soft sets $F_A$ and $G_B$ with $A \cap B \neq \emptyset$ over $(U, E)$, is the interval-valued hesitant fuzzy soft set $H_C$, where $C = A \cap B$, and, $\forall e^h C$, $\mu_{H(e)} = \mu_{F(e)^h} \cap \mu_{G(e)^h}$. We write $F_A \bigcap G_B = H_C$.

3. Interval-valued hesitant fuzzy soft topological space

In this section, first we give a counter example of equality of IVHFSSs proposed by Zhang et al. [32]. Secondly we point out that proposition 3.11 in a previous paper by Borah and Hazarika [4] true in general by counter example. Thirdly we introduce about notion of topological space.

Definition 3.1. [32] Let $F_A$ and $G_B$ be two interval-valued hesitant fuzzy soft sets. Now $F_A$ and $G_B$ are said to be interval-valued hesitant fuzzy soft equal if and only if

(i) $F_A \preceq G_B$, (ii) $G_B \preceq F_A$,

This can be denoted by $F_A = G_B$.

Example 3.2. Let,

$F_A = \{ e_1 = \{ < h_1, [0.2, 0.5] >, < h_2, [0.5, 0.8] >, [0.4, 0.9] > \},$

$e_2 = \{ < h_1, [0.3, 0.6], [0.4, 0.8] >, < h_2, [0.6, 0.8] > \}$.

$G_A = \{ e_1 = \{ < h_1, [0.2, 0.5], [0.2, 0.5] >, < h_2, [0.5, 0.8], [0.4, 0.9] > \},$

$e_2 = \{ < h_1, [0.3, 0.6], [0.4, 0.8] >, < h_2, [0.6, 0.8], [0.6, 0.8] > \}$.

Therefore

$F_A \preceq G_A$, and $G_A \preceq F_A$.

Hence

$F_A = G_A$.

Proposition 3.3. Let $F_A$, $G_B$ and $H_C$ be three interval-valued hesitant fuzzy soft sets. Then the following are satisfied:

(i) $F_A \bigcup (G_B \bigcap H_C) = (F_A \bigcup G_B) \bigcap (F_A \bigcap H_C)$,

(ii) $F_A \bigcap (G_B \bigcup H_C) = (F_A \bigcap G_B) \bigcup (F_A \bigcap H_C)$.

Proof. We consider IVHFSSs.

$F_A = \{ e_1 = \{ < h_1, [0.2, 0.5] >, < h_2, [0.3, 0.2], [0.5, 0.6], [0.3, 0.6] >,\}$

$e_2 = \{ < h_1, [0.2, 0.9], [0.7, 1.0] >, < h_2, [0.81, 1.0], [0.2, 0.6] > \}$.

$G_B = \{ e_1 = \{ < h_1, [0.7, 0.9], [0.0, 0.6] >, < h_2, [0.4, 0.7], [0.4, 0.5] >,\}$

$e_2 = \{ < h_1, [0.6, 0.8] >, < h_2, [0.3, 0.8], [0.3, 0.6] >\}$

$e_3 = \{ < h_1, [0.5, 0.6], [0.3, 0.6] >, < h_2, [0.1, 0.6], [0.3, 0.9], [0.3, 0.6] > \}$.

And
$H_C = \{e_2 = \{h_1, [0.4,0.6], [0.2,0.6], [0.7,1.0] >, h_2, [0.3,0.8] >\},
\}
e_3 = \{h_1, [0.2,0.5], [0.3,0.5] >, h_2, [0.6,0.8], [0.2,0.5] >\}.$

(i) We have

$F_A \cap G_B = \{e_1 = \{h_1, [0.3,0.8], [0.7,0.9] >, h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\}
e_2 = \{h_1, [0.6,0.9], [0.7,1.0] >, h_2, [0.3,0.6], [0.8,1.0] >\}
\}
e_3 = \{h_1, [0.3,0.6], [0.5,0.6] >, h_2, [0.1,0.6], [0.3,0.9], [0.3,0.6] >\}.

$F_A \cap H_C = \{e_1 = \{h_1, [0.3,0.8] >, h_2, [0.3,0.6], [0.3,0.8] >, [0.5,0.6] >,\}
\}
e_2 = \{h_1, [0.2,0.9], [0.7,1.0], [0.7,1.0] >, h_2, [0.3,0.6], [0.8,1.0] >\}
\}
e_3 = \{h_1, [0.2,0.5], [0.3,0.5] >, h_2, [0.1,0.5], [0.3,0.8], [0.3,0.6] >\}.$

$(F_A \cap G_B) \cap (F_A \cap H_C) = \{e_1 = \{h_1, [0.3,0.8], [0.3,0.8] >, h_2, [0.3,0.6], [0.3,0.8] >, [0.5,0.6] >,\}
\}
e_2 = \{h_1, [0.2,0.9], [0.7,1.0], [0.7,1.0] >, h_2, [0.3,0.6], [0.8,1.0] >\}
\}
e_3 = \{h_1, [0.2,0.5], [0.3,0.5] >, h_2, [0.1,0.5], [0.3,0.8], [0.3,0.6] >\}.$

$G_B \cap H_C = \{e_2 = \{h_1, [0.2,0.6], [0.4,0.6], [0.6,0.8] >, h_2, [0.3,0.6], [0.3,0.8] >,\}
\}
e_3 = \{h_1, [0.2,0.5], [0.3,0.5] >, h_2, [0.1,0.5], [0.3,0.8], [0.3,0.6] >\}.

Therefore,

$F_A \cap G_B \cap H_C = (F_A \cap G_B) \cap (F_A \cap H_C).$

(ii) We have,

$F_A \cap G_B = \{e_1 = \{h_1, [0.0,0.6], [0.3,0.8] >, h_2, [0.3,0.5], [0.3,0.7], [0.4,0.6] >,\}
\}
e_2 = \{h_1, [0.2,0.8], [0.6,0.8] >, h_2, [0.2,0.6], [0.3,0.8] >\}.$

$G_B \cap H_C = \{e_1 = \{h_1, [0.0,0.6], [0.7,0.9] >, h_2, [0.4,0.5], [0.4,0.7], [0.4,0.7] >,\}
\}
e_2 = \{h_1, [0.6,0.8], [0.6,0.8], [0.7,1.0] >, h_2, [0.3,0.8], [0.3,0.8] >\}
\}
e_3 = \{h_1, [0.3,0.6], [0.5,0.6] >, h_2, [0.2,0.6], [0.6,0.9], [0.6,0.8] >\}.$

Therefore,

$F_A \cap (G_B \cap H_C) = \{e_1 = \{h_1, [0.0,0.6], [0.3,0.8] >, h_2, [0.3,0.5], [0.3,0.7], [0.4,0.6] >,\}
\}
e_2 = \{h_1, [0.2,0.8], [0.6,0.8], [0.7,1.0] >, h_2, [0.2,0.6], [0.3,0.8] >\}.$

Again,

$F_A \cap H_C = \{e_2 = \{h_1, [0.2,0.6], [0.4,0.6], [0.7,1.0] >, h_2, [0.2,0.6], [0.3,0.8] >\}.$

Therefore,

$(F_A \cap G_B) \cap (F_A \cap H_C) = \{e_1 = \{h_1, [0.0,0.6], [0.3,0.8] >, h_2, [0.3,0.5], [0.3,0.7], [0.4,0.6] >,\}
\}
e_2 = \{h_1, [0.2,0.8], [0.6,0.8], [0.7,1.0] >, h_2, [0.2,0.6], [0.3,0.8] >\}.$

Hence $F_A \cap (G_B \cap H_C) = (F_A \cap G_B) \cap (F_A \cap H_C).$
Definition 3.4. A interval-valued hesitant fuzzy soft topology \( \tau \) on \((U, E)\) is a family of interval-valued hesitant fuzzy soft sets over \((U, E)\) satisfying the following properties:

(i) \( \tilde{\Phi}, \tilde{E} \subseteq \tau \)

(ii) \( F_A, G_B \subseteq \tau \) then \( F_A \cap \tilde{G}_B \subseteq \tau \).

(i) If \( F_{\alpha} \subseteq \tau \) for all \( \alpha \in \Delta \), then \( \cup_{\alpha \in \Delta} F_{\alpha} \subseteq \tau \).

Example 3.5. Let \( U = \{h_1, h_2\} \) and \( E = \{e_1, e_2, e_3\} \) and consider \( A = \{e_1, e_2, e_3\}, B = \{e_1, e_2\} \subseteq E \).

Let

\[
F_A = \{e_1 = \{<h_1, [0.7,0.9]>,<h_2, [0.4,0.6], [0.5,0.7], [0.4,0.8] >\}
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.3,0.9], [0.1,0.6] >\}
\]

\[
G_B = \{e_1 = \{<h_1, [0.3,0.8], <h_2, [0.3,0.8], [0.3,0.6] >\},
\]

\[
e_2 = \{<h_1, [0.2,0.9], [0.7,1.0], <h_2, [0.8,1.0], [0.2,0.6] >\}
\]

Now rearrange the membership value of \( F_A \) and \( G_B \) with the help of Definitions 2.9, 2.10 and assumptions given by [8], we have

\[
F_A = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\}
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.1,0.6], [0.3,0.9] >\}
\]

\[
G_B = \{e_1 = \{<h_1, [0.3,0.8], <h_2, [0.3,0.6], [0.3,0.8] >\},
\]

\[
e_2 = \{<h_1, [0.2,0.9], [0.7,1.0], <h_2, [0.2,0.6], [0.8,1.0] >\}
\]

Suppose a collection \( \tau \) of interval-valued hesitant fuzzy soft sets over \((U, E)\) as \( \tau = \{\tilde{\Phi}, \tilde{E}, \tilde{F}_A, \tilde{G}_B\} \).

Therefore

(i) \( \tilde{\Phi}, \tilde{E} \subseteq \tau \)

(ii) \( \tilde{\Phi} \cap \tilde{E} = \tilde{\Phi}, \tilde{\Phi} \cap F_A = \tilde{\Phi}, \tilde{\Phi} \cap G_B = \tilde{\Phi}, F_A \cap \tilde{E} = F_A, G_B \cap \tilde{E} = G_B \) and

\[
F_A \cap \tilde{G}_B = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\]

\[
e_2 = \{<h_1, [0.2,0.9], [0.7,1.0], <h_2, [0.2,0.6], [0.8,1.0] >\}
\]

Hence

\[
F_A \cap \tilde{G}_B = G_B \).
\]

(iii) \( \tilde{\Phi} \cap \tilde{E} = \tilde{E}, \tilde{\Phi} \cap F_A = F_A, \tilde{\Phi} \cap G_B = G_B, F_A \cap \tilde{E} = \tilde{E}, G_B \cap \tilde{E} = \tilde{E} \) and

\[
F_A \cap \tilde{G}_B = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.1,0.6], [0.3,0.9] >\}
\]

\[
F_A \cap \tilde{G}_B = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.1,0.6], [0.3,0.9] >\}
\]

\[
F_A \cap \tilde{G}_B = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.1,0.6], [0.3,0.9] >\}
\]

\[
F_A \cap \tilde{G}_B = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.1,0.6], [0.3,0.9] >\}
\]

\[
F_A \cap \tilde{G}_B = \{e_1 = \{<h_1, [0.3,0.8], [0.7,0.9], <h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\},
\]

\[
e_2 = \{<h_1, [0.6,0.9], [0.7,1.0], <h_2, [0.3,0.6], [0.8,1.0] >\}
\]

\[
e_3 = \{<h_1, [0.3,0.6], [0.5,0.6], <h_2, [0.1,0.6], [0.3,0.9] >\}
\]

Therefore \( \tau \) is a IVHFS topology on \((U, E)\).

Definition 3.6. If \( \tau \) is a IVHFS topology on \((U, E)\), the triple \((U, E, \tau)\) is said to be a interval-valued hesitant fuzzy soft topological space (IVHFSTS). Also each member of \( \tau \) is called a interval-valued hesitant fuzzy soft open set in \((U, E, \tau)\).

Example 3.7. From example 3.5, The triple \((U, E, \tau)\) is a IVHFS topological space and the interval-valued hesitant fuzzy soft open sets in \((U, E, \tau)\) are \( \tilde{\Phi}, \tilde{E}, \tilde{F}_A, \tilde{G}_B \).
We write if and only if its complement \( F_A^c \) is a interval-valued hesitant fuzzy soft open set in \((U, E, \tau)\).

Definition 3.9. Let \((U, E, \tau)\) be a IVHFSTS. Let \( F_A \) be a IVHFSS over \((U, E)\). The interval-valued hesitant fuzzy soft closure of \( F_A \) is defined as the intersection of all interval-valued hesitant fuzzy soft closed sets(IVHFSCS) which contained \( F_A \) and is denoted by \( cl(F_A) \) or \( \overline{F}_A \). We write

\[
cl(F_A) = \overline{\{ G_B : G_B \text{ is IVHFSS and } F_A \subseteq G_B \}}.
\]

Example 3.10. From example 3.5, we have

\[
F_A = \{ e_1 = \langle h_1, [0.3, 0.8], [0.7, 0.9] \rangle, < h_2, [0.4, 0.6], [0.4, 0.8], [0.5, 0.7] \rangle
\]
\[
e_2 = \langle h_1, [0.6, 0.9], [0.7, 1.0] \rangle, < h_2, [0.3, 0.6], [0.8, 1.0] \rangle
\]
\[
e_3 = \langle h_1, [0.3, 0.6], [0.5, 0.6], < h_2, [0.1, 0.6], [0.3, 0.9] \rangle \}.
\]

\[
G_B = \{ e_1 = \langle h_1, [0.3, 0.8] \rangle, < h_2, [0.3, 0.6], [0.3, 0.8] \rangle,
\]
\[
e_2 = \langle h_1, [0.2, 0.9], [0.7, 1.0] \rangle, < h_2, [0.2, 0.6], [0.8, 1.0] \rangle,
\]
\[
e_3 = \langle h_1, [0.0, 0.0], [0.0, 0.0], < h_2, [0.0, 0.0], [0.0, 0.0] \rangle \}.
\]

Then interval-valued hesitant fuzzy soft closed sets are

\[
F_A^c = \{ e_1 = \langle h_1, [0.1, 0.3], [0.2, 0.7] \rangle, < h_2, [0.3, 0.5], [0.2, 0.6], [0.4, 0.6] \rangle \}
\]
\[
e_2 = \langle h_1, [0.0, 0.3], [0.1, 0.4], < h_2, [0.0, 0.2], [0.4, 0.7] \rangle
\]
\[
e_3 = \langle h_1, [0.4, 0.5], [0.4, 0.7], < h_2, [0.1, 0.7], [0.4, 0.9] \rangle \}.
\]

\[
G_B^c = \{ e_1 = \langle h_1, [0.2, 0.7] \rangle, < h_2, [0.2, 0.7], [0.4, 0.7] \rangle,
\]
\[
e_2 = \langle h_1, [0.0, 0.3], [0.1, 0.8], < h_2, [0.0, 0.2], [0.4, 0.8] \rangle,
\]
\[
e_3 = \langle h_1, [1.0, 1.0], [1.0, 1.0], < h_2, [1.0, 1.0], [1.0, 1.0] \rangle \}.
\]

Suppose interval-valued hesitant fuzzy soft set \( I_C \) over \((U, E)\) as

\[
I_C = \{ e_1 = \langle h_1, [0.1, 0.2], [0.1, 0.7] \rangle, < h_2, [0.3, 0.4], [0.1, 0.6], [0.4, 0.5] \rangle \},
\]
\[
e_2 = \langle h_1, [0.0, 0.2], [0.1, 0.7] \rangle, < h_2, [0.0, 0.1], [0.4, 0.8] \rangle,
\]
\[
e_3 = \langle h_1, [0.0, 0.0], [0.0, 0.0], < h_2, [0.0, 0.0], [0.0, 0.0] \rangle \}.
\]

Then

\[
cl(I_C) = \overline{\langle C \rangle} \cap G_B^c = G_B^c = \{ e_1 = \langle h_1, [0.2, 0.7] \rangle, < h_2, [0.2, 0.7], [0.4, 0.7] \rangle,
\]
\[
e_2 = \langle h_1, [0.0, 0.3], [0.1, 0.8] \rangle, < h_2, [0.0, 0.2], [0.4, 0.8] \rangle,
\]
\[
e_3 = \langle h_1, [1.0, 1.0], [1.0, 1.0], < h_2, [1.0, 1.0], [1.0, 1.0] \rangle \}.
\]

Proposition 3.11. Let \((U, E, \tau)\) be a IVHFSTS and \( F_A, G_B \) be two IVHFSSs over \((U, E)\).

Then the following are true:

(i) \( cl(\tilde{\phi}) = \tilde{\phi}, cl(\overline{E}) = \overline{E} \).

(ii) \( F_A \subseteq cl(F_A) \)

(iii) \( F_A \) is an interval-valued hesitant fuzzy soft closed set iff \( F_A = cl(F_A) \).

(iv) \( F_A \subseteq G_B \Rightarrow cl(F_A) \subseteq cl(G_B) \).
(v) $\text{cl}(F_A \bigcup G_B) = \text{cl}(F_A) \bigcup \text{cl}(G_B)$.

(vi) $\text{cl}(F_A \bigcap G_B) \subseteq \text{cl}(F_A) \bigcap \text{cl}(G_B)$.

(vii) $\text{cl}(\text{cl}(F_A)) = \text{cl}(F_A)$.

Proof. (i) Obvious.

(ii) The proof directly follows from definition.

(iii) Let $(U, E, \tau)$ be a IVHFSTS. Let $F_A$ be a IVHFSS over $E$ such that $F_A = \text{cl}(F_A)$. Therefore from definition of interval-valued hesitant fuzzy soft closure, we have $\text{cl}(F_A)$ is interval-valued hesitant fuzzy soft closed sets. Hence $\text{cl}(F_A)$ is interval-valued hesitant fuzzy soft closed and $\text{cl}(F_A) = F_A$. i.e. $F_A$ is interval-valued hesitant fuzzy soft closed.

Conversely, let $F_A$ be interval-valued hesitant fuzzy soft closed in $(U, E, \tau)$. Therefore from definition of interval-valued hesitant fuzzy soft closure that any interval-valued hesitant fuzzy soft closed set $G_B$, $F_A \subseteq G_B \Rightarrow \text{cl}(F_A) \subseteq G_B$.

Since $F_A \subseteq F_A \Rightarrow \text{cl}(F_A) \subseteq F_A$ and from definition $F_A \subseteq \text{cl}(F_A)$.

Hence it follows that $F_A = \text{cl}(F_A)$.

(iv) Let $F_A \subseteq G_B$. Since $G_B \subseteq \text{cl}(G_B)$. Therefore $F_A \subseteq \text{cl}(G_B)$.

Again $\text{cl}(F_A)$ is the smallest interval-valued hesitant fuzzy soft closed set containing $F_A$.

Hence $\text{cl}(F_A) \subseteq \text{cl}(G_B)$.

(v) From definition of union of IVHFSSs

$F_A \subseteq F_A \bigcup G_B, G_B \subseteq F_A \bigcup G_B$.

Therefore $\text{cl}(F_A) \subseteq \text{cl}(F_A \bigcup G_B) \subseteq \text{cl}(F_A \bigcup G_B)$.

$\Rightarrow \text{cl}(F_A) \bigcup \text{cl}(G_B) \subseteq \text{cl}(F_A \bigcup G_B)$.

(A1)

Again $\text{cl}(F_A \bigcup G_B) \subseteq \text{cl}(F_A) \bigcup \text{cl}(G_B)$.

(A2)

Since $\text{cl}(F_A \bigcup G_B)$ is the smallest interval-valued hesitant fuzzy soft closed set containing $F_A \bigcup G_B$. Hence from (A1) and (A2),

$\text{cl}(F_A \bigcup G_B) = \text{cl}(F_A) \bigcup \text{cl}(G_B)$.

(vi) From definition of intersection of IVHFSSs

$F_A \bigcap G_B \subseteq F_A, F_A \bigcap G_B \subseteq G_B$.

Therefore

$\text{cl}(F_A \bigcap G_B) \subseteq \text{cl}(F_A), \text{cl}(F_A \bigcap G_B) \subseteq \text{cl}(G_B)$.

$\Rightarrow \text{cl}(F_A \bigcap G_B) \subseteq \text{cl}(F_A) \bigcap \text{cl}(G_B)$.

(vii) If $F_A$ is a interval-valued hesitant fuzzy soft closed set then $F_A = \text{cl}(F_A)$.

Hence $\text{cl}(\text{cl}(F_A)) = \text{cl}(F_A)$.

Definition 3.12. Let $(U, E, \tau)$ be a IVHFSTS. Let $F_A$ be a IVHFSS over $E$. The interval-valued hesitant fuzzy soft interior of $F_A$ is defined as the union of all interval-valued hesitant fuzzy soft open sets (IVHFSOSs) which contained $F_A$ and is denoted by $\text{int}(F_A)$. We write $\text{int}(F_A) = \bigcup \{ G_B : G_B \text{is IVHFSOS and } G_B \subseteq F_A \}$.

Example 3.13. From example 3.5, we consider a interval-valued hesitant fuzzy soft set $I_C$ over $(U, E)$ as
\[ I_C = \{ e_1 = \langle h_1,[0.3,0.8]>, \langle h_2,[0.3,0.7],[0.3,0.8] > \}, \]
\[ e_2 = \langle h_1,[0.2,0.9],[0.7,1.0]>, \langle h_2,[0.2,0.7],[0.8,1.0] > \}, \]
\[ e_3 = \langle h_1,[0.0,0.0],[0.0,0.0]>, \langle h_2,[0.0,0.0],[0.0,0.0] > \}. \]

Therefore
\[ \text{int}(I_C) = G_B \cap \tilde{\phi} = G_B = \]
\[ \{ e_1 = \langle h_1,[0.3,0.8]>, \langle h_2,[0.3,0.6],[0.3,0.8] > \}, \]
\[ e_2 = \langle h_1,[0.2,0.9],[0.7,1.0]>, \langle h_2,[0.2,0.6],[0.8,1.0] > \}, \]
\[ e_3 = \langle h_1,[0.0,0.0],[0.0,0.0]>, \langle h_2,[0.0,0.0],[0.0,0.0] > \}. \]

Proposition 3.14. Let \( (U,E,\tau) \) be a IVHFSTS and \( F_A, G_B \) be two IVHFSs over \( (U,E) \). Then the following are true:

(i) \( \text{int}(\tilde{\phi}) = \tilde{\phi}, \text{int}(\tilde{E}) = \tilde{E}. \)

(ii) \( \text{int}(F_A) \subseteq F_A \)

(iii) \( F_A \) is an interval-valued hesitant fuzzy soft open set iff \( F_A = \text{int}(F_A) \).

(iv) \( F_A \subseteq G_B \Rightarrow \text{int}(F_A) \subseteq \text{int}(G_B) \).

(v) \( \text{int}(F_A) \cap \text{int}(G_B) \subseteq \text{int}(F_A \cap G_B) \).

(vi) \( \text{int}(\text{int}(F_A)) = \text{int}(F_A) \).

Proof. (i) Obvious.

(ii) The proof directly follows from definition.

(iii) Let \( (U,E,\tau) \) be a IVHFSTS. Let \( F_A \) be a IVHFSS over \( (U,E) \) such that \( F_A = \text{int}(F_A) \). Therefore from definition of interval-valued hesitant fuzzy soft interior, we have \( \text{int}(F_A) \) is interval-valued hesitant fuzzy soft open sets. Hence \( \text{int}(F_A) \) is interval-valued hesitant fuzzy soft open and \( \text{int}(F_A) = F_A \), i.e., \( F_A \) is interval-valued hesitant fuzzy soft open.

Conversely, let \( F_A \) be interval-valued hesitant fuzzy soft open in \( (U,E,\tau) \). Therefore from definition of interval-valued hesitant fuzzy soft interior that any interval-valued hesitant fuzzy soft open set \( G_B \triangleleft F_A \Rightarrow G_B \subseteq \text{int}(F_A) \).

Since \( F_A \triangleleft F_A \Rightarrow F_A \subseteq \text{int}(F_A) \) and from definition \( \text{int}(F_A) \subseteq F_A \).

Hence it follows that \( F_A = \text{int}(F_A) \).

(iv) Let \( F_A \triangleleft G_B \) since \( \text{int}(F_A) \subseteq F_A \triangleleft G_B \), therefore \( \text{int}(F_A) \) be an interval valued hesitant fuzzy soft open subset of \( G_B \).

Hence from definition of interval valued hesitant fuzzy soft interior, we have
\[ F_A \triangleleft G_B \Rightarrow \text{int}(F_A) \triangleleft \text{int}(G_B) \]

(v) Since
\[ F_A \triangleleft F_A \cap G_B \cap G_B \triangleleft F_A \cap G_B \]

Therefore we have \( \text{int}(F_A) \triangleleft \text{int}(F_A \cap G_B) \), \( \text{int}(G_B) \triangleleft \text{int}(F_A \cap G_B) \).

Hence
\[ \text{int}(F_A) \cap \text{int}(G_B) \triangleleft \text{int}(F_A \cap G_B) \]

(vi) Since
\[ F_A \cap G_B \triangleleft F_A \cap G_B \triangleleft G_B \]

These implies that
\[ \text{int}(F_A \cap G_B) \triangleleft \text{int}(F_A \cap G_B) \triangleleft \text{int}(G_B) \]

Therefore
\[ \text{int}(F_A \cap G_B) \triangleleft \text{int}(F_A) \cap \text{int}(G_B) \triangleleft \text{int}(G_B) \]
Again we know that
\[ \text{int}(F_A) \subseteq F_A \text{ and } \text{int}(G_B) \subseteq G_B. \] Therefore
\[ \text{int}(F_A) \cap \text{int}(G_B) \subseteq F_A \cap G_B \] (B2)
Hence from (B1) and (B2) we get
\[ \text{int}(F_A) \cap G_B = \text{int}(F_A) \cap \text{int}(G_B). \]

(vii). From (iii), if \( F_A \) is an interval-valued hesitant fuzzy soft open set then \( \text{int}(F_A) = F_A \). Therefore \( \text{int}(\text{int}(F_A)) = \text{int}(F_A) \).

Proposition 3.15. If \( \{ \tau_\alpha : \lambda \in I \} \) is a family of IVHFSTs on \( (U, E) \), then \( \cap \lambda \{ \tau_\alpha : \lambda \in I \} \) is also an IVHFST on \( (U, E) \).

Proof. Suppose \( \{ \tau_\alpha : \lambda \in I \} \) be a IVHFST. Therefore \( \tilde{\phi}, \tilde{E} \subseteq \cap \lambda \{ \tau_\alpha \} \). If \( F_A, G_B \subseteq \cap \lambda \{ \tau_\alpha \} \) then
\[ F_A \cap G_B \subseteq \tau_\alpha, \forall \lambda \in I. \] Therefore \( F_A \cap G_B \subseteq \tau_\alpha, \forall \lambda \in I \).

Thus \( F_A \subseteq \tau_\alpha \subseteq \cap \lambda \{ \tau_\alpha \} \).

Let \( \{ F_A \}_{\alpha \in J} \subseteq \cup \lambda \{ \tau_\alpha \} \).

Therefore \( F_A \subseteq \tau_\alpha \subseteq \cup \lambda \{ \tau_\alpha \} \).

Definition 3.16. Let \( \tau_1 \) and \( \tau_2 \) be IVHFSTs on \( (U, E) \). We say that \( \tau_1 \) is coarser (or weaker) than \( \tau_2 \) or \( \tau_2 \) is finer (or stronger) than \( \tau_1 \) if and only if \( \tau_1 \subseteq \tau_2 \) i.e. every \( \tau_1 \) intervalvalued hesitant fuzzy soft open set \( (IVHFOS) \) is \( \tau_2 \) IVHFOS. Again \( \tau_1 \) and \( \tau_2 \) are said to be comparable if either \( \tau_1 \subseteq \tau_2 \) or \( \tau_2 \subseteq \tau_1 \). If \( \tau_1 \not\subseteq \tau_2 \) and \( \tau_2 \not\subseteq \tau_1 \), then we say the IVHFST \( \tau_1 \) and \( \tau_2 \) are not comparable.

Example 3.17. From example 3.5, we consider IVHFST \( \tau_1 \) and \( \tau_2 \) on \( (U, E) \) as
\[ \tau_1 = (\tilde{\phi}, \tilde{E}, \tilde{F}_A), \tau_2 = (\tilde{\phi}, \tilde{E}, \tilde{F}_A, \tilde{G}_B). \]

Therefore \( \tau_1 \subseteq \tau_2 \) and hence \( \tau_1 \) is coarser than \( \tau_2 \).
Definition 3.22. A IVHFSS $I_C$ in a IVHFSTS $(U, E, \tau)$ is called a interval valued hesitant fuzzy soft neighbourhood (IVHFSNBD) of the IVHFSP $e(F_A) \subseteq (U, E)$ if there is a IVHFSOS $G_B$ such that $e(F_A) \subseteq G_B \subseteq I_C$.

Example 3.23. From examples 3.19, 3.21, we consider the IVHFST $\tau = \{\tilde{\phi}, \tilde{E}, \tilde{G}_B\}$, and IVHFSS $I_C$ as

$I_C = \{e_1 = \{<h_1,[0.2,1.0]>, <h_2,[0.2,0.3]>, \ldots\}\}$

Then for each $E \subseteq G_B \subseteq I_C$. Therefore $e(F_A) \subseteq G_B \subseteq I_C$.

Hence $I_C$ is a IVHFSNHD of the IVHFSP $e_2(F_A)$.

Definition 3.24. The family consisting of all neighbourhoods of $e(F_A) \subseteq (U, E)$ neighbourhood system of a fuzzy soft point $e(F_A)$. It is denoted by $N_r(e(F_A))$.

Definition 3.25. A IVHFSS $I_C$ in a IVHFSTS $(U, E, \tau)$ is called a IVHFSNBD of the IVHFSS $H_A$ if there is a IVHFSOS $G_B$ such that $H_A \subseteq G_B \subseteq I_C$.

Example 3.26. From examples 3.21, 3.23 and consider the IVHFSS $H_A$ as

$H_A = \{e_1 = \{<h_1,[0.1,0.5]>, <h_2,[0.6,0.7]>, \ldots\}\}$

Then there is an IVHFSOS $H_A \subseteq G_B \subseteq I_C$.

Hence IVHFSS $I_C$ is IVHFSNBD of the IVHFSS $H_A$.

Proposition 3.27. The neighbourhood system $N_r(e(F_A))$ at $\forall e(F_A)$ in an IVHFSTS $(U, E, \tau)$ has the following properties:

(i) If $G_B \subseteq N_r(e(F_A))$ then $e(F_A) \subseteq G_B$.

(ii) If $G_B \subseteq N_r(e(F_A))$ and $G_B \subseteq H_C$ then $H_C \subseteq N_r(e(F_A))$.

(iii) If $G_B, H_C \subseteq N_r(e(F_A))$ then $G_B \cap H_C \subseteq N_r(e(F_A))$.

(iv) If $G_B \subseteq N_r(e(F_A))$ then there is a $H_C \subseteq N_r(e(F_A))$ such that $G_B \subseteq N_r(e(M_D))$ for each $e(M_D) \subseteq H_C$.

Proof. (i) If $G_B \subseteq N_r(e(F_A))$, then there is a IVHFSOS $H_C$ such that $e(F_A) \subseteq H_C \subseteq G_B$.

Therefore we have $e(F_A) \subseteq G_B$.

(ii) Let $G_B \subseteq N_r(e(F_A))$ and $G_B \subseteq H_C$. Then there is a $L_D$ such that $e(F_A) \subseteq L_D \subseteq G_B$ and $e(F_A) \subseteq L_D \subseteq G_B \subseteq H_C$. Therefore $H_C \subseteq N_r(e(F_A))$.

(iii) If $G_B, H_C \subseteq N_r(e(F_A))$ then there exist IVHFSOSs $L_D, M_E$ such that $e(F_A) \subseteq L_D \subseteq G_B$ and $e(F_A) \subseteq M_E \subseteq H_C$. Thus $e(F_A) \subseteq L_D \cap M_E \subseteq G_B \cap H_C$. Since $L_D \cap M_E \subseteq \tau$. Hence we have $G_B \cap H_C \subseteq N_r(e(F_A))$.

(iv) If $G_B \subseteq N_r(e(F_A))$, then there is an IVHFSOS $L_D \subseteq \tau$ such that $e(F_A) \subseteq L_D \subseteq G_B$.

Now put $H_C = L_D$. Then for each $e(M_D) \subseteq H_C = L_D \subseteq G_B$. This implies...
\[ G_B \subseteq \mathcal{N}(e'(M_D)). \]

References