Connell Sequences

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Abstract:- In this paper we have discussed an interesting type of sequence called "Connell Sequence". The main part of the paper lies in deriving the generating function for the Connell Sequence and its limiting behavior is discussed. We have extended the definition of Connell Sequence and have studied the "General Connell Sequences" and have studied its generating function and its limiting behavior. The connection between Connell Sequences and the Polygonal Numbers are also explored in the process of finding the Generating Functions of these Sequences.

Keywords: *Sequences, Subsequences, Limiting Behaviour, Generating Functions, Polygonal Numbers*

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1. Introduction:

In 1959 Ian Connell **[1]** defined a curious sequence which now bears his name. The sequence is given by

1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, 19, 21, 23, 25, ...
(A001614 in the On-Line Encyclopedia of Integer Sequences).

It is created by the following way: The first odd number is followed by the next two even numbers, which in turn are followed by the next three odd numbers, and so on.

Lakhtakia and Pickover **[2]** have studied the interesting generating function for n given by
$C(n) = 2n- [(1+ Sqrt(8n-7))/2]$, where [x] is the greatest integer less than or equal to x and Sqrt (x) is the Square root of x. From the above formula they have also asserted that C(n)/n is 2 for large n.

As a first step we will now demonstrate a new proof of generating function for C(n) and subsequently the behavior of C(n)/n as n approaches infinity.

For establishing the generating function for C(n) we first observe that the above sequence can be concatenated in to finite subsequences,

Subsequence Number:	Subsequence
1	1
2	2,4
3	5,7,9
4	10,12,14,16
...	...

where the nth subsequence contains n elements, the last of which is n^2. So if we let C(n) denote the nth element of the Connell Sequence then

$C(T_n) = n^2$...(1), where T_n is the nth Triangular Number given by n(n+1)/2.

Now $|S_1| + |S_2| + ... +|S_n| = 1+2+3+...+n = T_n$ and so the last element of S_n is $C(T_n)$. Thus S_{n+1} begins with the element $C(T_n) +1$ and since $|S_{n+1}| = n+1$, it ends with the element $C(T_n) + 1+ 2n$. But this last element is also expressible as $C(T_{n+1})$. Therefore, assuming the induction hypothesis $C(T_n) = n^2$, we obtain

$C(T_{n+1}) = C(T_n) + 1 + 2n = n^2 + 1 + 2n = (n+1)^2$. Hence by induction hypothesis we have for all positive integers n, $C(T_n) = n^2$. Here T_n is the nth Triangular number and n^2 is the nth Square number. Thus C expresses relationship between Triangular and Square numbers. Using this we will now derive the generating function for C(n).
Define the sequence H by H(n) = 2n- C(n) ...(2)

Let n be a positive integer. There is a positive j, and fixed i such that $1 \leq i \leq 1+j$, for which n = T(j)+i.

Thus C(n) belong to the Subsequence S_{j+1}. As C(T(j)) is the last element of S_j , we have

C(n) = C(T(j)+i) = C(T(j))+1+2(i-1)...(3)
Now, H(n) = 2n-C(n) = 2n - {C(T(j))+1+2(i-1)}
$= 2n-\{j^2+1+2i-2\} = 2\{j(j+1)/2+i\}-\{j^2+2i-1\} = j+1$.
Hence H(n) = j+1...(4)

Since $n \geq T(j)+1$, $n-1 \geq T(j) = j(j+1)/2$ and so we have $j^2+j-2(n-1) \leq 0$, which is a quadratic inequality in j, and so we have $j \leq (-1+\text{Sqrt}(8n-7))/2$ and $j+1 \leq (1+ \text{Sqrt}(8n-7))/2$, so from (4) we have

H(n) = [(1+ Sqrt(8n-7))/2] and from (2) we get

C(n) = 2n-[(1+ Sqrt(8n-7))/2] which is the Generating Function for the Connell Sequence C(n) defined above. Now, C(n)/n = 2-[1/2n+0.5√(8/n)-(7/ n^2)] and since (a/n) approaches zero as n approaches infinity, the ratio C(n)/n approaches 2 as n is very large. Thus $\lim C(n)/n = 2$ as $n \to \infty$.

2. Generalized Connell Sequence with parameters

For fixed integers $m \geq 2$ and $r \geq 1$ we construct a sequence as follows: Take the first integer which is congruent to 1 (mod m) (that being 1 itself), followed by next 1+r integers which are congruent to 2 (mod m), followed by 1+2r integers which are congruent to 3 (mod m), and so on. If m=2 and r=1 (the smallest possible cases) we have the Connell Sequence discussed above. Here is a formal definition.

Definition 1: Let $m \geq 2$ and $r \geq 1$ be integers. We denote by $C_{m,r}$ (n) the nth term of the generalized Connell Sequence with parameters m and r, or, simply the
Connell (m,r)-Sequence. The sequence is defined as follows:

1. The sequence is formed by concatenating subsequences S_1, S_2, ..., each of finite length.
2. The subsequence S_1 consists of element 1.
3. If the nth subsequence S_n ends with the element e, then the (n+1)th subsequence S_n+1 begins with the element e+1.
4. If S_n consists of t elements, then S_{n+1} consists of t+r elements.
5. Each subsequence is non-decreasing, and the difference between two consecutive elements in the same subsequence is m.

From the above definition the Connell Sequence discussed in **1.** is $C_{2,1}(n)$. As another example we can consider the general Connell Sequence $C_{3,2}(n)$ whose elements according to the above definition are given by 1,2,5,8,9,12,15,18,21,22,25,28,31,34,37,40,...

If we follow the above definition we can arrange the above sequence in terms of subsequences as follows:

n	S_n
1	1
2	2,5,8
3	9,12,15,18,21
4	22,25,28,31,34,37,40
...	...

The final elements 1,8,21,40, ... in the subsequences appear to be the Octagonal numbers, $E_n = n(3n-2)$. The nth subsequence contains exactly 2n-1 elements, and from the identity $1+3+5+...+(2n-1) = n^2$ we obtain the relation

$C_{3,2}(n^2) = E_n$. Just as the Connell Sequence relates triangular numbers to squares, the sequence $C_{3,2}(n)$ relates Squares to Octagonal numbers. Triangular numbers, Squares and Octagonal numbers are all examples of Polygonal numbers. So there exist a definite relation between $C_{m,r}(n)$ and the polygonal numbers for each m and r.

3. Relationships with Polygonal Numbers

Definition 2: For integers $k \geq 3$, the nth k-gonal number is defined as

$P_k(n) = n\{(k-2)n-(k-4)\}/2$ **.**

We shall demonstrate the relationship between generalized Connell Sequences and Polygonal Numbers **[3]**.
S_1 contains 1 element and S_2 contains 1+r elements, S_3 contains 1+2r elements and so on. Therefore S_n contains exactly 1+(n-1)r elements.(Refer Conditions 2 and 4 in the Definition given above). Therefore to reach the end of nth subsequence S_n , we must count exactly
$|S_1| + |S_2| + ... +|S_n| = 1+(1+r)+(1+2r)+...+1+(n-1)r = n\{2+(n-1)r\}/2 = P_{r+2}(n)$ (By definition 2) elements of the sequence $C_{m,r}(n)$. Hence the last element of S_n is $C_{m,r}(P_{r+2}(n))$. Thus S_{n+1} begins (Refer Condition 3 of Definition 1) with the element $C_{m,r}(P_{r+2}(n))+1$, and since,
$|S_{n+1}| = 1+nr$, it ends with (Refer condition 5 of Definition 1) the element $C_{m,r}(P_{r+2}(n))+1+mnr$. But this last element is also expressible as $C_{m,r}(P_{r+2}(n+1))$. Therefore assuming the Induction Hypothesis $C_{m,r}(P_{r+2}(n)) = P_{mr+2}(n)$, we obtain
$C_{m,r}(P_{r+2}(n+1)) = C_{m,r}(P_{r+2}(n))+1+mnr = P_{mr+2}(n)+1+mnr = P_{mr+2}(n+1)$. (By definition **2**)

Thus $C_{m,r}(P_{r+2}(n+1)) = P_{mr+2}(n+1)$, and hence by induction we have for all positive integers n, $C_{m,r}(P_{r+2}(n)) = P_{mr+2}(n)$, which is the relation between generalized Connell Sequences and Polygonal numbers.

As an illustration we find that $C_{2,1}(P_3(n)) = P_4(n)$
i.e. $C_{2,1}(T_n) = n^2$, since $P_3(n)$ is T_n a Triangular number and $P_4(n)$ is a Square number. Similarly $C_{3,2}(P_4(n)) = P_8(n)$ i.e. $C_{3,2}(n^2) = E_n$, where $P_8(n) = E_n$ is a Octagonal number, which were established before.

4. Limiting Behavior

We will determine the behavior of $C_{m,r}(n)/n$ as n approaches infinity, using the relation
$C_{m,r}(P_{r+2}(n)) = P_{mr+2}(n)$ derived above.

Let n be a positive integer. There is a positive j, and a fixed i such that $1 \leq i \leq 1+rj$, for which $n = P_{r+2}(j) + i$. Thus $C_{m,r}(n) = C_{m,r}(P_{r+2}(j) + i) = C_{m,r}(P_{r+2}(j))+1+m(i-1)$, is the ith element of the subsequence S_{j+1}. As $C_{m,r}(P_{r+2}(j))$ is the last element of S_j, from Definition **1**, we have
$C_{m,r}(n) = C_{m,r}(P_{r+2}(j))+1+m(i-1) = P_{mr+2}(j)+1+m(i-1)$.
Now since $n = P_{r+2}(j) + i$ and $1 \leq i \leq 1+rj$ we have
$P_{mr+2}(j)+1 \leq P_{mr+2}(j)+1+m(i-1) \leq P_{mr+2}(j)+1+mrj$ and
$P_{r+2}(j)+1 \leq P_{r+2}(j) + i \leq P_{r+2}(j) + 1+rj$ and so we get
$(P_{mr+2}(j)+1)/(P_{r+2}(j) + 1+rj) \leq (P_{mr+2}(j)+1+m(i-1))/(P_{r+2}(j) + i) \leq (P_{mr+2}(j)+1+mrj)/(P_{r+2}(j)+1)$
i.e. $A \leq C_{m,r}(n)/n \leq B$...(5), where
$A = (P_{mr+2}(j)+1)/(P_{r+2}(j) + 1+rj)$ and $B = (P_{mr+2}(j)+1+mrj)/(P_{r+2}(j)+1)$. Now using definition **2**. we see that both A and B approaches m as $j \to \infty$ and since $n = P_{r+2}(j) + i$,
$n \to \infty$ as $j \to \infty$. Therefore from (5), by Squeeze Principle, we have $C_{m,r}(n)/n$ also approaches m as $n \to \infty$. Thus
$\lim C_{m,r}(n)/n = m$ as $n \to \infty$.

5. Generating Function for $C_{m,r}(n)$

To find a generating function for $C_{m,r}(n)$ we modify Korsak's **[4]** proof which is used in **1**. for deriving the generating function for $C_{2,1}(n)$.

Define the sequence H by $H(n) = mn - C_{m,r}(n)$...(6)

we assume n>1and $n = P_{r+2}(j) + i$ exactly as in **4**. Now, from (6) we have $H(n) = mn - C_{m,r}(n)$

$H(n) = mn – \{C_{m,r}(P_{r+2}(j))+1+m(i-1)\}$

$H(n) = mn – \{ P_{mr+2}(j)+1+m(i-1)\}$

$= mn-1-im+m-j\{mrj-(mr-2)\}/2$

$= mj\{rj-(r-2)\}/2+mi-1-im+m-j\{mrj-(mr-2)\}/2$

$= mj+m-j-1 = (m-1)(j+1)$ and so

$H(n) = (m-1)(j+1)$ from which $j+1 = H(n)/(m-1)$...(7)

Since $n \geq P_{r+2}(j) +1$, we have $n-1 \geq j\{rj-(r-2)\}/2$ and so

$r j^2 – (r-2)j – 2(n-1) \leq 0$, which is a quadratic inequality in j. So from the above inequality we get

$j \leq \{(r-2) + \mathrm{Sqrt}((r-2)^2 +8r(n-1))\}/2r$ where Sqrt(x) is the Square root of x. Therefore we have

$j+1 \leq \{(3r-2) + \mathrm{Sqrt}((r-2)^2 +8r(n-1))\}/2r$ and since j+1 is always an integer we have

$j+1 = [\{(3r-2) + \mathrm{Sqrt}((r-2)^2 +8r(n-1))\}/2r]$, where [x] is the greatest integer $\leq$ x. Thus from (6) and (7) we have

$C_{m,r}(n) = mn – (m-1) [\{(3r-2) + \mathrm{Sqrt}((r-2)^2 +8r(n-1))\}/2r]$,

which is the generating function of $C_{m,r}(n)$.

As a special case if we put m = 2 and r = 1 we get the generating function of $C_{2,1}(n)$ which is given by

$C_{2,1}(n) = 2n – [(1 + \mathrm{Sqrt}(8n-7))/2]$ which is derived in **1**.

References

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