Generalized Pigeon Hole Principle and its Applications

R. Sengothai
Ph.D. Research Scholar in Mathematics
University of Madras
Email: kothai1729@yahoo.com
Contact Number: 9382878023

Abstract:- In this paper I shall introduce "The Pigeon Hole Principle" in usual way and then present and prove the general versions of the Pigeon Hole Principle, hereby referred as PHP. I shall introduce several applications of the above mentioned principle by solving some examples.

Keywords: *Principle, Pigeons, Holes, Generalized Pigeon Hole Principle*

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PIGEONHOLE PRINCIPLE (PHP)

If m pigeons occupy n pigeonholes and m>n , then at least 1 pigeonhole must contain more than 1 pigeon.

GENERALIZATIONS OF PHP

1. "If m pigeons occupy n pigeon holes, then at least 1 pigeonhole must contain (p+1) or more pigeons where p = [(m-1)/n]."

PROOF

We prove this principle by the method of contradiction.
Assume that the conclusion part of the principle is not true. Then no pigeonhole contains (p+1) or more pigeons. This means that every pigeonhole contains p or less number of pigeons.
Thus, the total number of pigeons is less than or equal to np= n * [(m-1)/n] which is less than or equal to n * ((m-1)/n)=(m-1) This is a contradiction ,because the total number of pigeons is m Hence our assumption is wrong., and the principle is true.

2. "Suppose m=(p_1 + p_2 + ………+p_n –n+1) pigeons occupy n pigeonholes H_1,H_2,……,H_n Prove that some pigeonhole H_j contains p_j or more pigeons."

PROOF

Assume that the conclusion part of the given statement is false. Then every hole H_j contains p_j-1 or less number of pigeons, j= 1,2,…n. Then the total number of pigeons would be less than or equal to (p_1-1) + (p_2-1)+……..+ (p_n-1) = (p_1+p_2+….p_n-n)= m-1. This is a contradiction, because the number of pigeons is equal to m. Hence the assumption made is wrong , and the given statement is true.

PROBLEMS

1. ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least 2 of these points are such that the distance between them is less than 0.5 cm
   Solution: consider triangle DEF formed by the mid-points of the sides BC, CA and AB of the given triangle ABC. Then the triangle ABC is partitioned into 4 small equilateral triangles, each of which has sides equal to 1/2 cm.
   Treating each of these portions as pigeonhole and 5 points chosen inside the triangle as pigeons, we find by using the PHP that at least 1 portion must contain 2 or more points. Evidently, the distance between such points is less than 1/2 cm.

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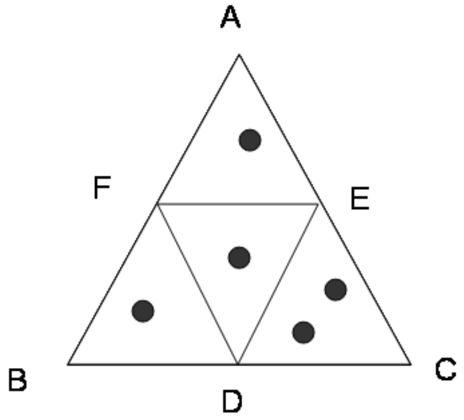
2. Prove that in any set of 29 persons at least 5 persons must have been born on the same day of the week.
   Solution: treating the 7 days of a week as 7 pigeonholes and 29 persons as 29 pigeons, we find by using the generalized PHP that at least 1 day of the week is assigned to [(29-1)/7] + 1 = 5 or more persons. In other words, at least 5 of any 29 persons must have been born on the same day of the week.



3. How many persons must be chosen in order that at least 5 of them will have birth days in the same calendar month?
   Solution: Let n be the required number of persons. Since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays in the same month is by the generalized PHP , equal to [(n-1)/12]+1.But this number is 5 if
   [(n-1)/12]+1 = 5 or n = 49.
   Thus the number of persons is at least 49.
4. Show that if any 5 numbers from 1 to 8 are chosen, then 2 of them will have their sum = 9
   Solution: Let us consider the following sets
   A_1={1,8}
   A_2={2,7}
   A_3={3,6}
   A_4={4,5}
   These are the only sets containing 2 numbers from 1 to 8 , whose sum is 9. Since every no. from 1to 8 belongs to 1 of the above sets ,each of the 5 numbers chosen must belong to 1 of the sets . Since there are only 4 sets , 2 of the 5 chosen numbers have to belong to the same set(according to PHP).
   These 2 numbers have their sum = 9.

5. Prove that every set of 37 positive integers contains atleast 2 integers that leave the same reminder upon division by 36.
   Solution: When a positive integer is divided by 36 , the possible reminders are 0,1,2….35. Let A_r denote the set of all positive integers that leave the reminder r when divided by 36.Thus , every positive integer belongs to one or the other of the 36 sets: A_0,A_1,…..A_{35}.Hence if we take any 37 positive integers then atleast 2 of them must belong to 1 of these A_r's.

   (Note: treat Ar's as pigeonholes and 37 as the number of pigeons).This proves the result.

6. Show that every set of 7 distinct integers includes 2 integers x and y such that atleast 1 of (x+y) or (x-y) is divisible by 10.
   Solution: Let X={x_1,x_2,…..,x_7} be a set of 7 distinct integers and let r_i be the reminder when x_i is divided by 10.

Consider the following subsets of X:
A_1={x_i belonging to X such that ri=0}
A_2={x_i belonging to X such that r_i=5 }
A_3={x_i belonging to X such that r_i=1 or 9}
A_4={x_i belonging to X such that r_i=2 or 8}
A_5={x_i belonging to X such that r_i=3 or 7}
A_6={x_i belonging to X such that r_i=4 or 6}
Now , the 7 elements of X play the role of pigeons and the 6 subsets listed above play the role of pigeonholes. As such atleast 2 elements x,y of X

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must be in the same subset .

If x and y are in A_1 then x and y are multiples of 10 so that both x +y x-y are multiples of 10. If x and y are in A_2 then x and y are of the forms $x = 10k_1 + 5$ and $y = 10k_2 + 5$ where k_1 and k_2 are integers, so that $x + y = 10(k_1+k_2+1)$ and $x-y=10(k_1-k_2)$ are both multiples of 10.

If x and y are in any of the other 4 subsets , then it is easily seen that either x-y or x+y is a multiple of 10, but not both. This proves the result.

8. Prove that if 101 integers are selected from the set S={1,2,3,……,200}, then at least 2 of these are such that one divides the other.

   Solution: Let X={1,3,5,…199}.then every integer between 1 and 200 (inclucive) is of the form n = (2^k)*x where k is an integer ≥ 0 and x belongs to X. Thus the set X has 100 distinct elements and therefore, if 101 elements of S are selected, then atleast 2 of them say a and b,a different from b must correspond to the same x belonging to x. Thus ,
   a= (2^m)*x , b=(2^n)*x , for some integers m,n ≥ 0 . Clearly, a divides b if m is < or = n and b divides a if n < m. This proves the required result.

9. Prove the statement: if m= kn + 1 pigeons (where k $\geq$1) occupy n pigeonholes then atleast 1 pigeonhole must contain k+1 or more pigeons.

   Solution: assume that the conclusion part of the given statement is false.then every pigeonhole contains k or less number of pigeons. Then , the total number of pigeons would be nk . This is a contradiction . Hence, the assumption made is wrong, and the given statement is true.

10. Prove that in a set of 13 children atleast 2 have birthdays during the same month.

    Solution: Let us treat the 13 children as pigeons(m=13) and the 12 months as 12 pigeon holes (n=12) . clearly m > n .hence by the PHP , atleast 1 month has 2 or more children 's birth days in it. This implies that atleast 2 children have birthdays during the same month .Hence the proof.

11. If 5 colours are used to paint 26 doors, show that atleast 6 doors will have the same colour.

    Let us treat the 26 doors as 26 pigeons (m=26) and the 5 doors as 5 pigeonholes (n=5) . Then by generalized PHP atleast 1 door will have
    [(m-1)/n]+1 =[(26-1)/5]+1 = 6 or more doors corresponding to it. This the same as saying that at least 6 doors will have the same colour. Hence the proof.

12. Prove that if 30 dictionaries in a library contain a total of 61327 book pages, then atleast 1 of the dictionaries must have atleast 2045 pages.

    Solution: treating the pages as pigeons and dictionaries as pigeonholes, we find by using the generalized PHP that atleast 1 of the dictionaries must contain (p+1) or more pages where
    p = [(61327-1)/30]= [2044.2]=2044
    hence the proof.

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