A Theorem on the Prime Graph of 2×2 - Matrix Ring of $\mathbf{Z}_2$

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Abstract:-In this paper we consider prime graph of R (denoted by $PG(R)$) of an associative ring R (introduced bySatyanarayana, Syam Prasad and Nagaraju [5]).This short paper is divided into two sections. Section-1 is devoted for preliminary definitions. In section -2we constructed prime graph of R(that is, PG(R)) where R = the set of all 2×2 matrices over $\mathbf{Z}_2$ and proved that it is a star graph. At the end some fundamental properties of PG(R) were listed.

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1.1 Introduction

Let G = (V, E) be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices {v_i, v_j}, where v_i, v_j are called end points of e_k. The edge $e_k$is also denoted by either $v_i v_j$ or $\overline{v_i v_j}$. We also write $G(V,E)$ for the graph. Vertex set and edge set of G are also denoted by $V(G)$ and $E(G)$ respectively. An edge associated with a vertex pair {v_i, v_i} is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and δ(v) denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loop or parallel edges is called a simple graph. We consider simple graphs only. For an associative ring R, prime graph ofR(denoted by $PG(R)$) was introduced in Satyanarayana, Syam Prasad and Nagaraju [5].

1.2Definitions:(i) A graph G (V, E) is said to be a star graph if there exists a fixed vertex v such that E = {vu / u $\in$ V and u $\neq$ v}. A star graph is said to be an n-star graph if the number of vertices of the graph is n.

(ii)Two graphs G and G^1 are said to be ***isomorphic*** if there is an one-to-one correspondence f between their vertices and an one-to-one correspondence g between their edges such that the incidence relationship must be preserved.
[In other words, two graphs G = (V, E) & G^1 = (V^1, E^1) are said to be ***isomorphic***if there exist bijections f: V $\rightarrow$ V^1 and g: E $\rightarrow$ E^1 such that g($\overline{v_i v_j}$) = $\overline{f(v_i) f(v_j)}$ for any edge $\overline{v_i v_j}$ in G].

(iii) In a graph G, a subset S of V(G) is said to be a ***dominating set*** if every vertex not in S has a neighbour in S. The ***domination number***, denoted by γ(G) is defined as min{|S| / S is a dominating set in G}.(iv)In a connected graph, a closed walk running through every vertex of G exactly once (except the starting vertex at which the walk terminates) is called as **Hamiltonian circuit.** A graph containing a Hamiltonian circuit is called as **Hamiltonian graph.**

1.3 Theorem: (Th. 13.8, page 361, [3]) A given connected graph G is an Eulerian graph if and only if all the vertices of G are of even degree.
For other fundamental concepts we refer [2], [3], [4] or [5]

2. The prime graph

2.1 Definition(Satyanarayana, Syam Prasad and Nagaraju[5]):A graph G(V, E) is said to be the**prime graph** of R (denoted by PG(R)) if V = R and E = { $\overline{xy}$ / xRy = 0 or yRx = 0, and x $\neq$ y}.

2.2 Theorem: If R = Set of all 2×2 matrices with the elements from the field $\mathbf{Z}_2$, then PG(R) is a 16-star graph.

Proof: It is clear that $|R|$=16. Consider $V\left(PG(R)\right)=$

$$\left\{\begin{array}{l} v_0=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ v_1=\begin{pmatrix}0&0\\0&1\end{pmatrix},\ v_2=\begin{pmatrix}0&0\\1&0\end{pmatrix}, v_3=\begin{pmatrix}0&0\\1&1\end{pmatrix},\ v_4=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ v_5=\begin{pmatrix}0&1\\0&1\end{pmatrix},\\ v_6=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ v_7=\begin{pmatrix}0&1\\1&1\end{pmatrix}, v_8=\begin{pmatrix}1&0\\0&0\end{pmatrix}, v_9=\begin{pmatrix}1&0\\0&1\end{pmatrix}, v_{10}=\begin{pmatrix}1&0\\1&0\end{pmatrix}, v_{11}=\begin{pmatrix}1&0\\1&1\end{pmatrix},\\ v_{12}=\begin{pmatrix}1&1\\0&0\end{pmatrix}, v_{13}=\begin{pmatrix}1&1\\0&1\end{pmatrix}, v_{14}=\begin{pmatrix}1&1\\1&0\end{pmatrix}, v_{15}=\begin{pmatrix}1&1\\1&1\end{pmatrix}\end{array}\right\}$$

Since $v_0 R v_i=0$, if $1\le i\le 15$ there is an edge between v_0 and v_i for all $1\le i\le 15$. Since $v_1v_3v_2\ne 0$, there is no edge between v_1 and v_2. Similarly there are no other edges between v_i & v_j if $i\ne j$ and $1\le i\le 15$. This conclusion is because of the following. Verification: Since $v_1v_3v_2\ne 0, v_1v_5v_3\ne 0, v_1v_3v_4\ne 0, v_1v_2v_5\ne 0, v_1v_5v_6\ne 0,\ v_1v_5v_7\ne 0,$ $v_1v_3v_8\ne 0,\ v_1v_2v_9\ne 0, v_1v_2v_{10}\ne 0, v_1v_2v_{11}\ne 0, v_1v_2v_{12}\ne 0, v_1v_2v_{13}\ne 0, v_1v_2v_{14}\ne 0, v_1v_2v_{15}\ne 0$

This part shows that there is no edge between v_1 and v_j (for $2\le j\le 15$). Now we observe the following.

$v_2v_4v_1\ne 0, v_2v_4v_3\ne 0, v_2v_8v_4\ne 0, v_2v_4v_5\ne 0, v_2v_4v_6\ne 0, v_2v_4v_7\ne 0, v_2v_9v_8\ne 0, v_2v_4v_9\ne 0,$
$v_2v_4v_{10}\ne 0, v_2v_4v_{11}\ne 0, v_2v_8v_{12}\ne 0, v_2v_4v_{13}\ne 0, v_2v_4v_{14}\ne 0, v_2v_4v_{15}\ne 0 v_3v_4v_1\ne 0, v_3v_4v_2\ne 0,$
$v_3v_2v_4\ne 0, v_3v_2v_5\ne 0, v_3v_2v_6\ne 0, v_3v_2v_7\ne 0, v_3v_2v_8\ne 0, v_3v_1v_9\ne 0, v_3v_2v_{10}\ne 0, v_3v_2v_{11}\ne 0,$
$v_3v_2v_{12}\ne 0, v_3v_1v_{13}\ne 0, v_3v_1v_{14}\ne 0, v_3v_2v_{15}\ne 0, v_4v_1v_1\ne 0, v_4v_1v_2\ne 0, v_4v_1v_3\ne 0, v_4v_2v_5\ne 0,$
$v_4v_2v_6\ne 0, v_4v_2v_7\ne 0, v_4v_2v_8\ne 0, v_4v_1v_9\ne 0, v_4v_2v_{10}\ne 0, v_4v_2v_{11}\ne 0, v_4v_2v_{12}\ne 0, v_4v_2v_{13}\ne 0,$
$v_4v_1v_{14}\ne 0, v_4v_2v_{15}\ne 0,\ v_5v_1v_1\ne 0, v_5v_1v_2\ne 0, v_5v_1v_3\ne 0, v_5v_2v_4\ne 0, v_5v_2v_6\ne 0, v_5v_2v_7\ne 0,$
$v_5v_2v_8\ne 0, v_5v_1v_9\ne 0, v_5v_2v_{10}\ne 0, v_5v_2v_{11}\ne 0, v_5v_2v_{12}\ne 0, v_5v_2v_{13}\ne 0, v_5v_1v_{14}\ne 0, v_5v_2v_{15}\ne 0,$
$v_6v_1v_1\ne 0, v_6v_1v_2\ne 0, v_6v_1v_3\ne 0, v_6v_2v_4\ne 0, v_6v_2v_5\ne 0, v_6v_2v_7\ne 0, v_6v_2v_8\ne 0, v_6v_1v_9\ne 0,$
$v_6v_2v_{10}\ne 0, v_6v_2v_{11}\ne 0, v_6v_2v_{12}\ne 0, v_6v_2v_{13}\ne 0, v_6v_2v_{14}\ne 0, v_6v_2v_{15}\ne 0,\ v_7v_1v_1\ne 0, v_7v_1v_2\ne 0,$
$v_7v_1v_3\ne 0, v_7v_2v_4\ne 0, v_7v_2v_5\ne 0, v_7v_2v_6\ne 0, v_7v_2v_8\ne 0, v_7v_2v_9\ne 0, v_7v_2v_{10}\ne 0, v_6v_2v_{11}\ne 0,$
$v_7v_2v_{12}\ne 0, v_7v_2v_{13}\ne 0, v_7v_2v_{14}\ne 0, v_7v_2v_{15}\ne 0, v_8v_4v_1\ne 0, v_8v_4v_2\ne 0, v_8v_4v_3\ne 0, v_8v_8v_4\ne 0,$
$v_8v_4v_5\ne 0, v_8v_4v_6\ne 0, v_8v_4v_7\ne 0, v_8v_4v_9\ne 0, v_8v_4v_{10}\ne 0, v_8v_4v_{11}\ne 0, v_8v_8v_{12}\ne 0, v_8v_4v_{13}\ne 0,$
$v_8v_4v_{14}\ne 0, v_8v_4v_{15}\ne 0 v_9v_1v_1\ne 0, v_9v_1v_2\ne 0, v_9v_1v_3\ne 0, v_9v_2v_4\ne 0, v_9v_1v_5\ne 0, v_9v_1v_6\ne 0,$
$v_9v_1v_7\ne 0, v_9v_2v_8\ne 0, v_9v_1v_{10}\ne 0, v_9v_1v_{11}\ne 0, v_9v_2v_{12}\ne 0, v_9v_1v_{13}\ne 0, v_9v_1v_{14}\ne 0, v_9v_1v_{15}\ne 0,$
$v_{10}v_4v_1\ne 0, v_{10}v_4v_2\ne 0, v_{10}v_4v_3\ne 0, v_{10}v_8v_4\ne 0, v_{10}v_4v_5\ne 0, v_{10}v_4v_6\ne 0, v_{10}v_4v_7\ne 0, v_{10}v_8v_8\ne 0,$
$v_{10}v_4v_9\ne 0, v_{10}v_4v_{11}\ne 0, v_{10}v_8v_{12}\ne 0, v_{10}v_4v_{13}\ne 0, v_{10}v_4v_{14}\ne 0, v_{10}v_1v_{15}\ne 0 v_{11}v_1v_1\ne 0, v_{11}v_1v_2\ne 0,$
$v_{11}v_1v_3\ne 0, v_{11}v_2v_4\ne 0, v_{11}v_1v_5\ne 0, v_{11}v_1v_6\ne 0, v_{11}v_1v_7\ne 0, v_{11}v_2v_8\ne 0, v_{11}v_1v_9\ne 0, v_{11}v_1v_{10}\ne 0,$
$v_{11}v_2v_{12}\ne 0, v_{11}v_1v_{13}\ne 0, v_{11}v_2v_{14}\ne 0, v_{11}v_1v_{15}\ne 0, v_{12}v_1v_1\ne 0, v_{12}v_1v_2\ne 0, v_{12}v_1v_3\ne 0, v_{12}v_2v_4\ne 0,$
$v_{12}v_1v_5\ne 0, v_{12}v_1v_6\ne 0, v_{12}v_1v_7\ne 0, v_{12}v_2v_8\ne 0, v_{12}v_1v_9\ne 0, v_{12}v_1v_{10}\ne 0, v_{12}v_1v_{11}\ne 0, v_{12}v_1v_{13}\ne 0,$
$v_{12}v_1v_{14}\ne 0, v_{12}v_1v_{15}\ne 0, v_{13}v_1v_1\ne 0, v_{13}v_1v_2\ne 0, v_{13}v_1v_3\ne 0, v_{13}v_2v_4\ne 0, v_{13}v_1v_5\ne 0, v_{13}v_1v_6\ne 0,$
$v_{13}v_1v_7\ne 0, v_{13}v_2v_8\ne 0, v_{13}v_1v_9\ne 0, v_{13}v_1v_{10}\ne 0, v_{13}v_1v_{11}\ne 0, v_{13}v_2v_{12}\ne 0, v_{13}v_1v_{14}\ne 0, v_{13}v_1v_{15}\ne 0,$ Now we got
$v_{14}v_1v_1\ne 0, v_{14}v_1v_2\ne 0, v_{14}v_1v_3\ne 0, v_{14}v_2v_4\ne 0, v_{14}v_1v_5\ne 0, v_{14}v_1v_6\ne 0, v_{14}v_1v_7\ne 0, v_{14}v_2v_8\ne 0,$
$v_{14}v_1v_9\ne 0, v_{14}v_1v_{10}\ne 0, v_{14}v_1v_{11}\ne 0, v_{14}v_2v_{12}\ne 0, v_{14}v_1v_{13}\ne 0, v_{14}v_1v_{15}\ne 0, v_{15}v_1v_1\ne 0, v_{15}v_1v_2\ne 0,$
$v_{15}v_1v_3\ne 0, v_{15}v_2v_4\ne 0, v_{15}v_1v_5\ne 0, v_{15}v_1v_6\ne 0, v_{15}v_1v_7\ne 0, v_{15}v_2v_8\ne 0, v_{15}v_1v_9\ne 0, v_{15}v_1v_{10}\ne 0,$
$v_{15}v_1v_{11}\ne 0, v_{15}v_2v_{12}\ne 0, v_{15}v_1v_{13}\ne 0, v_{15}v_1v_{14}\ne 0.$

that E(PG(R)) = $\left\{\overline{v_o v_j} / 1 \le j \le 15\right\}$. Therefore PG(R) is a 16-star graph. Since the vertex v_0 dominates the graph, we have that the domination number of PG(R) is 1.

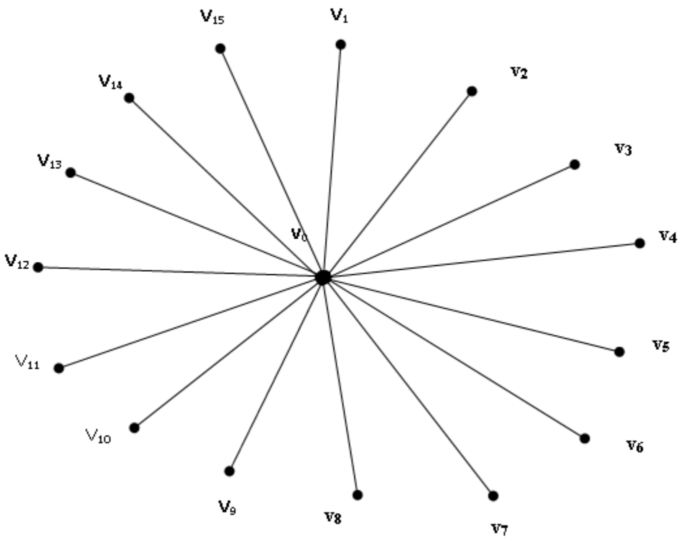

Fig. 2. 4 (Graph PG(R))

2.3 Corollary: The prime graph of the 2×2 - matrix ring of $\mathbb{Z}_2$ is a 16 – star graph; and any 16 – star graph is isomorphic to the PG(R) where R is the 2×2 - matrix ring of $\mathbb{Z}_2$.

2.4 Observations: Consider the graph PG(R) where R is the set of all 2×2 matrix over $\mathbb{Z}_2$

(i). The graph is shown in Fig. 2.4
(ii). The graph contains no cycles, Hence it is a bipartite graph
(iii). The vertex v_0 dominates the graph and so the domination number of PG(R) is one
(iv). This is a planar graph
(v). Chromatic number of PG(R) is 2.
(vi). PG(R) is not Eulerian graph(Th. 13.8, page 361, [3])
(vii). PG(R) is not a Hamiltonian graph.

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