A Study on Primitive Ring

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Abstract: In this paper we discus about prime ring, simple ring, primitive ring and some lemmas and theorems.

Key words: simple ring, Prime ring, Reduced Primitive ring, division ring.

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1. INTRODUCTION

Throughout the paper, the word 'ring' means an associative ring (but not necessarily commutative ring with an identity element 1). The word 'subring' always means a subring containing the identity element of the larger ring. If R= {0}, R is called the zero ring, it is noted that this is the case if and only if 1= 0 in R. If R ≠ {0} and ab=0 implies either a= 0 or b= 0, R is said to be domain and if R is commutative then R is called an integral domain. Without exception the word 'ideal' refers to a 2- sided ideal. First we give an example of an element x in a ring R such that Rx $\not\subset$ xR. Let R be the ring of 2×2 upper triangular matrices over a nonzero ring k and let x= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then Rx= $\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ and xR= $\begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$, therefore Rx $\not\subset$ xR.

A ring R is called prime if aRb= (0) implies either a= 0 or b= 0. So a ring R is a prime if 0 is prime ideal of R. Example of a prime ring, for integer n ≠ 0, R= $\begin{pmatrix} Z & nZ \\ Z & Z \end{pmatrix}$ is a prime ring. It is a subring of S= $M_2(\mathbf{Z})$. Note that nS ⊂ R. If a, b ∈ R such that aRb= 0. Since S is a prime ring. We conclude that a=0 or b= 0, we know that a ring R is prime if and only if $M_n(\mathbb{R})$ is prime, where $M_n(\mathbb{R})$ is the set of n × n matrices with entries from $\mathbb{R}$.

A nonzero ring R is said to be a simple if (0) and R are the only ideal in R. A ring R is said to be a reduced ring if R has no nonzero nilpotent elements or equivalently if a^2= 0 implies a= 0. For instance, the direct product of any family of domain is reduced. An ideal generated by a single element is called a principal ideal. For example the trivial ideal 0 and the ideal R are both principal because 0= (0) and R= (1).

2. DEFINITION

An ideal P of a ring R is called left primitive if it is the largest ideal of R contained in some maximal left ideal M of R. Thus P= [R: M]= { r ∈ R : rR ⊂ M}. This definition is not symmetrical, and it is known that left primitive does not imply right primitive. None the less; the attribute left is usually omitted. A ring is called primitive if 0 is a primitive ideal. It easily follows that, for any ideal P, $\frac{R}{P}$ is a primitive ring if and only if P is primitive ideal.

Let R be a ring. A left R- module is an additive abelian group M together with an action (called scalar multiplication) μ: R×M→ M(the image μ(r, x) being denoted by rx) such that for all r, s ∈ R and x, y ∈ M:

(i) r (x+ y)= rx+ ry
(ii) (r+ s) x= rx+ sx
(iii) r (sx)= (rs) x

It is denoted by R^M. Let M be an R- module . A non empty subset N of M is called an R- submodule of M if and only if

(i) x, y ∈ N implies x- y ∈ N
(ii) x ∈ N and r ∈ R implies rx ∈ N.

Let R, k be two rings and M = R^{Vk} be an (R, k) – bi-module . We write E= End (V_k), which operates on V from the left. We say that R acts densely on V_k if, for any f ∈ E and any $v_1, v_2, . . ., v_n$ ∈ V, there exists r ∈ R such that rv_i= f (v_i) for i= 1, 2, . . . ,n. A module M (≠{0}) is said to be simple if and only if the only submodule of M are {0} and M.

A left R- module R^A is called faithful if whenever rA= 0 implies r= 0, r ∈ R. In other words A is faithful if for any 0 ≠ r, r ∈ R, rA ≠ 0.

3. SOME LEMMAS AND THEOREMS

3.1 Lemma: A ring R with identity is a domain if and only if R is prime and reduced.

Proof: Assume a ring R is a domain. Then a^n= 0 implies a= 0, so R is reduced. Also if aRb=0 implies ab= 0 which implies a= 0 or b= 0. So R is a prime.

Conversely, assume R is prime and reduced. Let a, b ∈ R such that ab= 0. Then for any r ∈ R, $(bra)^2$= brabra= br(ab) ra= 0, so bra= 0. This means that bRa= 0, so b= 0 or a= 0 since R is prime.

3.2 Lemma: Every simple ring is a primitive ring.

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Proof: Let R be a simple ring. Then 0 is a maximal ideal of R. Let M be a maximal left ideal of R Then 0 is the largest ideal contained in M. Hence 0 is a primitive ideal of R which shows that R is primitive ring.

3.3 Lemma: Every maximal ideal is a primitive ideal.

Proof: Let I be a maximal ideal of R. Then $\frac{R}{I}$ is a simple ring which implies that $\frac{R}{I}$ is a primitive ring since every simple ring is primitive which implies that I is a primitive ideal of R.

3.4 Lemma: A commutative ring is primitive if and only if it is a field.

Proof: Given R is a commutative ring. Assume R is a primitive ring. Then 0 is a primitive ideal of R which implies that 0 is the largest ideal of R contained in some maximal left ideal M of R since R is a commutative. So we have 0= [R: M], where [R: M]= { a ∈ R : aR ⊆ M}, since P is a primitive ideal of R if and only if P= [R :M]

Now let a ∈ M. Then aR ⊆ M implies a ∈ [R: M]. But [R: M]= 0, so a= 0.

Therefore 0=M implies 0 is a maximal ideal of R. Since R is commutative ring. So R is a field.

Conversely, Assume R is a field. We know that a field has no proper ideals. So 0 is a primitive ideal of R which implies that R is a primitive ring.

3.4 Theorem: Let P be a proper ideal of a commutative ring R. Then P is a prime if and only if whenever aRb ⊆ P implies a ∈ P or b ∈ P, a, b ∈ R.

Proof: Since P is a proper ideal of R. Assume that P is a prime and aRb ⊆ P (a, b ∈ R). Then

(RaR) (RbR) ⊆P, since P is a two sided ideal & RaR is a principal ideal of R generating by a ∈ P. This implies RaR ⊆ P or RbR ⊆ P, since P is prime. So 1.a.1 ∈ P or 1.b.1 ∈ P which implies a ∈ P or b ∈ P

Conversely, Let A and B be two ideals of R such that AB ⊆ P. If possible A ⊄ P, then there exists a ∈ A such that a ∉ P. But aRb ⊆AB ⊆ P for all b ∈ B, which implies aRb ⊆ P for all b ∈ B.

Since a ∉ P so we have b ∈ P implies B⊆ P. So P is a prime ideal of R

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3.5 Lemma: A primitive ring is prime.

Proof: Let R be a primitive ring. So 0 is a primitive ideal. Let aRb= 0. There exists a faithful simple left R- module A. Then RbA is a submodule of A. Hence RbA= 0 or RbA= A, since A is simple. If RbA= 0 then bA= 0 if R= 1 which implies b= 0, since A is faithful. If RbA= A, then aRbA= aA but aRb= 0. Therefore aA= 0 implies a= 0 since A is faithful. This implies that 0 is a prime ideal of R, so R is a prime ring.

3.6 Corollary: A primitive ideal of a ring R is prime ideal.

Proof: Let P be a primitive ideal of a ring R. Then $\frac{R}{P}$ is a primitive ring which implies that $\frac{R}{P}$ is prime ring. This implies P is a prime ideal of R.

3.7. [2] Density Theorem: Let R be a ring and V be a semi-simple left R module. Then for k= End (R^V), R acts densely on V_k.

3.8. Theorem: Let R be a left primitive ring such that a (ab- ba)= (ab- ba) a for all a, b ∈ R. Then R is a division ring.

Proof: Let R^V be a faithful simple R- module with k= End (R^V). It suffices to show that dim $_R$V= 1 (For then R ≅ k which is a division ring). Assume instead there exist k- linearly independent vectors u, v ∈ V. By density theorem there exist a, b ∈ R such that au= u, av= 0 and bu= 0, bv= u. But then a (ab- ba) (v)= a^2u= u, (ab- ba) a (v)= 0, a contradiction.

3.9. Theorem: Let R be a left primitive ring such that 1+ a^2 is a unit for any a ∈ R. Then R is a division ring.

Proof: We repeat the argument. If the independent vectors u, v ∈ V exist, we can find a ∈ R such that a (u)= -v and a (v)= u. Then (1+ a^2) (u)= u- u= 0 which contracts the assumption.

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