Stochastic Multifacility Location Problem under Triangular Area Constraint with Euclidean Norm

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Abstract:- The multifacility location issue is an augmentation of the single-location problem in which we might be keen on finding the location of various new facilities concerning different existing locations. In the present study, multifacility location under triangular zone limitation with probabilistic methodology for the weights considered in the objective function and the Euclidean distances between the locations has been presented. Scientific detailing and the explanatory arrangement have been acquired by utilizing Kuhn-Tucker conditions. The arrangement strategy has been represented with the assistance of a numerical illustration. Two sub-instances of the issue in each of which the new locations are to be situated in semi-open rectangular zone have likewise been talked about.

Key words:- *multifacility, constraint,stochastic, location problem, Euclidean and Kuhn-Tucker conditions.*

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1. Introduction

Investigations in the past have mainly been done without taking into account the availability of the area into which the new locations fall.In this regard we may mention the work of various authors cited under references [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,20,21,22,23,24,25]. But Santra and Nasira [18, 19] have considered the deterministic model of the multifacility location problem under triangular and semi-open rectangular area constraints. The present investigation is the stochastic version of our previous work.The problem of multifacility location under triangular area restriction with probabilistic approach for the weights considered in the objective function is studied. Physically such probabilistic approach is of great importance in the sense that the values of weights from various origins (sources) to different destinations are not the fixed quantities but they take random values in different situations. This necessity in tackling these sorts of problems has motivated to study the stochastic version of the multifacility location problem. I have considered the problem in which the weights considered in the objective function are the random variables with discrete probabilities and the distance between the facilities is Euclidean. For the present investigation, the interactions among new facilities as well as between new and existing facilities have been considered. Two sub-cases of the problem in each of which the new facilities are to be located in semi-open rectangular area have also been presented.

2. Formulation and Solution Procedures of the Problem

The stochastic multifacility location problem considered with triangular area constraint can be stated as:

Minimize $f((x_1, y_1), (x_2,y_2),\ldots\ldots\ldots, (x_n,y_n))$

$$= \sum_{1\le j<k\le n} E(v_{jk})\left[(x_j - x_k)^2 + (y_j - y_k)^2 + \varepsilon\right]^{1/2} + \sum_{i=1}^{m}\sum_{j=1}^{n} E(w_{ji})\left[(x_j - \hat{x}_i)^2 + (y_j - \hat{y}_i)^2 + \varepsilon\right]^{1/2} \ldots (1)$$

Subject to $ax_j + by_j + c \le 0$, $(a, b>0, c<0)$ … (2)
and $x_j \ge 0$, $y_j \ge 0$ ($j = 1,2,\ldots\ldots\ldots,n$)
where n = number of new facilities,
m= number of existing facilities,
(x_j,y_j)= co-ordinates of the j[th] new facility
$(\hat{x}_i, \hat{y}_i)$= co-ordinates of the i[th] existing facility,
$E(v_{jk})$ = expected value of the cost per unit distance between new facility j and new facility k, $E(v_{jj}) = 0$,
$E(w_{ji})$ = expected value of the cost per unit distance between new facility j and existing facility i,
and the expected values of v_{jk} and w_{ji} are defined as follows:

$$E(v_{jk}) = \sum_{\alpha_{(jk)}=1}^{N_{(jk)}} p_{(jk)\,\alpha_{(jk)}} \cdot v_{(jk)\,\alpha_{(jk)}} \quad ; \quad \sum_{\alpha_{(jk)}=1}^{N_{(jk)}} p_{(jk)\,\alpha_{(jk)}} = 1 \quad ;$$

$$\text{and} \quad E(w_{ji}) = \sum_{\beta_{(ji)}=1}^{M_{(ji)}} q_{(ji)\,\beta_{(ji)}} . w_{(ji)\,\beta_{(ji)}} \; ; \quad \sum_{\beta_{(ji)}=1}^{M_{(ji)}} q_{(ji)\,\beta_{(ji)}} = 1.$$

The objective function as well as side constraints is non-linear. Kuhn-Tucker conditions are used to get the solutions of the problem for which it is needed to construct the auxiliary function as follows:

$h(x,y) = f(x,y) - \lambda[(ax + by + c) + \eta^2]$ … (3)

i.e., $h((x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n))$

$= f((x1,y1), (x_2,y_2), \ldots, (x_n,y_n)) - \sum_{j=1}^{n} \lambda_j [(ax_j + by_j + c) + \eta_j^2]$… (4)

where η_j^2 are the artificial variables and are given by

$$\eta_j^2 = -(ax_j + by_j + c).$$

Now by using Kuhn-Tucker theory we get the following set of necessary conditions:

$$\frac{\partial h}{\partial x_j} = \frac{\partial f}{\partial x_j} - a\lambda_j = 0 \quad ,\ldots (5)$$

$$\frac{\partial h}{\partial y_j} = \frac{\partial f}{\partial y_j} - b\lambda_j = 0 \quad , \quad \ldots (6)$$

$\lambda_j (ax_j + by_j + c) = 0$ …(7)

$ax_j + by_j + c \leq 0.$ …. (8)

and $\lambda_j = 0$ $(j = 1,2,\ldots,n)$.

We may note that the necessary conditions for the occurrence of the minimum are sufficient in view of the convexity of the objective function. In view of (1), we get

$$\frac{\partial f}{\partial x_j} = \sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})(x_j - x_k)}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})(x_j - \hat{x}_i)}{E_{ji}} \quad \ldots (9)$$

$$\text{and} \quad \frac{\partial f}{\partial y_j} = \sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})(y_j - y_k)}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})(y_j - \hat{y}_i)}{E_{ji}} \ldots (10)$$

where $D_{jk} = [(x_j - x_k)^2 + (y_j - y_k)^2 + \varepsilon]^{1/2}$ … (11)

$E_{ji} = [(x_j - \hat{x}_i)^2 + (y_j - \hat{y}_i)^2 + \varepsilon]^{1/2}$ … (12)

and $v'_{jk} = \begin{cases} v_{jk}, & k > j \\ v_{kj}, & k < j \end{cases}$ … (13)

In view of Kuhn-Tucker conditions we are to examine only two possible cases, viz., (i) when $\lambda_j = 0$ and (ii) when $\lambda_j \neq 0$ $(j = 1,2,\ldots,n)$. We consider first the case when $\lambda_j = 0$.

2.1 Case – I $\lambda_j = 0$ $(j = 1, 2,\ldots, n)$

Since $\lambda_j = 0$, the equations (5) and (6) reduce *to* $\frac{\partial f}{\partial x_j} = 0$ and $\frac{\partial f}{\partial y_j} = 0$ respectively, which in view of (9) and (10) lead respectively to

$$\sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})(x_j - x_k)}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})(x_j - \hat{x}_i)}{E_{ji}} = 0 \ldots (14)$$

and

$$\sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})(y_j - y_k)}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})(y_j - \hat{y}_i)}{E_{ji}} = 0 \ldots (15)$$

After straightforward calculation (14) and (15) lead respectively to

$$x_j = \frac{\sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})x_k}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})\hat{x}_i}{E_{ji}}}{\sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})}{E_{ji}}} \quad \ldots (16)$$

and

$$y_j = \frac{\sum_{\substack{j=1 \\ \neq j}}^{n} \frac{E(v'_{jk})y_k}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})\hat{y}_i}{E_{ji}}}{\sum_{\substack{k=1 \\ \neq j}}^{n} \frac{E(v'_{jk})}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})}{E_{ji}}} , \quad (j = 1, 2, \ldots, n) \quad \ldots (17)$$

To solve the set of non-linear equations represented by (16) and (17), we use the following iterative scheme:

The equations (16) and (17) can be written as:

$x_j = F_j(x_1,x_2,\ldots, x_n,y_1,y_2,\ldots,y_n)$ … (18)

and $y_j = G_j(x_1, x_2, \ldots\ldots, x_n, y_1, y_2, \ldots\ldots, y_n)$ ... (19)

Starting with initial solution $(x_1^{(0)}, x_2^{(0)}, \ldots\ldots, x_n^{(0)}, y_1^{(0)}, y_2^{(0)}, \ldots\ldots, y_n^{(0)})$ we form the successive approximate solutions $(x_1^{(1)}, x_2^{(1)}, \ldots\ldots, x_n^{(1)}, y_1^{(1)}, y_2^{(1)}, \ldots\ldots, y_n^{(1)})$, $(x_1^{(2)}, x_2^{(2)}, \ldots\ldots, x_n^{(2)}, y_1^{(2)}, y_2^{(2)}, \ldots\ldots, y_n^{(2)})$, ……. , $(x_1^{(N)}, x_2^{(N)}, \ldots\ldots, x_n^{(N)}, y_1^{(N)}, y_2^{(N)}, \ldots\ldots, y_n^{(N)})$ from the relation given by

$x_j^{(N+1)} = F_j(x_1^{(N)}, x_2^{(N)}, \ldots\ldots, x_n^{(N)}, y_1^{(N)}, y_2^{(N)}, \ldots\ldots, y_n^{(N)})$ … (20)

and $y_j^{(N+1)} = G_j(x_1^{(N)}, x_2^{(N)}, \ldots\ldots, x_n^{(N)}, y_1^{(N)}, y_2^{(N)}, \ldots\ldots y_n^{(N)})$ …(21)

where the superscripts denote the iteration number. We take the initial solution as:

$$x_j^{(0)} = \frac{\sum_{i=1}^{m} E(w_{ji})\,\hat{x}_i}{\sum_{\substack{k=1\\ \neq j}}^{n} E(v'_{jk}) + \sum_{i=1}^{m} E(w_{ji})} \quad \ldots (22)$$

and

$$y_j^{(0)} = \frac{\sum_{i=1}^{m} E(w_{ji})\,\hat{y}_i}{\sum_{\substack{k=1\\ \neq j}}^{n} E(v'_{jk}) + \sum_{i=1}^{m} E(w_{ji})}, \quad (j = 1, 2, \ldots\ldots, n). \quad \ldots (23)$$

for the rapid convergence of the iterative scheme. The scheme will be convergent provided the following inequality holds:

$$\left|\frac{\partial F_j(t_{j1}, t_{j2}, \ldots\ldots, t_{j2n})}{\partial x_k}\right| < \frac{M}{n} \text{ for all j and k,}$$

where $0 < M < 1$ and $t_{jk} = x_k^{(N)} + (\beta_k - x_k^{(N)})\,\theta_j$, $0 < \theta_j < 1$,
($k = 1,2,\ldots,n, n+1,\ldots, 2n$; $j = 1,2,\ldots\ldots,n,n+1,\ldots\ldots,2n$) and where $x_{n+1}, x_{n+2}, \ldots\ldots, x_{2n}$ stand for $y_1, y_2, \ldots\ldots y_n$ respectively, and β_j are the solution of (18).

The solution (x_j, y_j) $(j = 1,2,\ldots\ldots,n)$ obtained by the above iterative scheme has to be tested whether it satisfies the constraint (2). If it satisfies the constraint (2), the problem is solved and (x_j, y_j) $(j = 1,2,\ldots\ldots,n)$ give the optimum location for the new facilities sought. If (x_j, y_j) $(j = 1,2,\ldots\ldots,n)$ do not satisfy the constraint (2), as already discussed, we have the only alternative of considering the case when $\lambda_j \neq 0$ $(j = 1,2,\ldots, n)$. In this case all the new facilities lie on the boundary.

2.2 Case – II $\lambda_j \neq 0$ $(j = 1,2,\ldots, n)$

Since $\lambda_j \neq 0$, the equation (7) takes the form $ax_j + by_j + c = 0$, which gives

$$y_j = -\frac{c}{b} - \frac{a}{b}x_j \text{ or } \quad y_k = -\frac{c}{b} - \frac{a}{b}x_k. \quad \ldots (24)$$

The equations (5) and (6) lead to

$$\frac{\partial f}{\partial x_j} = \frac{a}{b}\frac{\partial f}{\partial y_j} \ldots (25)$$

and this in view of (9) and (10) takes the form:

$$\sum_{\substack{k=1\\ \neq j}}^{n} \frac{E(v'_{jk})(x_j - x_k)}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})(x_j - \hat{x}_i)}{E_{ji}} = \frac{a}{b}\left[\sum_{\substack{k=1\\ \neq j}}^{n} \frac{E(v'_{jk})(y_j - y_k)}{D_{jk}} + \sum_{i=1}^{m} \frac{E(w_{ji})(y_j - \hat{y}_i)}{E_{ji}}\right] \quad \ldots (26)$$

Further simplification of (26) by using (11) and (12) and the value of y_j or y_k from (24) finally lead to

$$x_j = \frac{\frac{b}{\sqrt{a^2+b^2}}\sum_{\substack{k=1\\ \neq j}}^{n} \frac{E(v'_{jk})\,x_k}{\sqrt{(x_j - x_k)^2 + \varepsilon\frac{b^2}{a^2+b^2}}} + \frac{b^3}{a^2+b^2}\sum_{i=1}^{m}\frac{E(w_{ji})\hat{x}_i}{T_{ji}} - \frac{ab^2}{a^2+b^2}\sum_{i=1}^{m}\frac{E(w_{ji})\hat{y}_i}{T_{ji}} - \frac{abc}{a^2+b^2}\sum_{i=1}^{m}\frac{E(w_{ji})}{T_{ji}}}{\frac{b}{\sqrt{a^2+b^2}}\sum_{\substack{k=1\\ \neq j}}^{n}\frac{E(v'_{jk})}{\sqrt{(x_j - x_k)^2 + \varepsilon\frac{b^2}{a^2+b^2}}} + b\sum_{i=1}^{m}\frac{E(w_{ji})}{T_{ji}}} \quad \ldots (27)$$

where $T_{ji} = \left[b^2(x_j - \hat{x}_i)^2 + (ax_j + b\hat{y}_i + c)^2 + \varepsilon b^2\right]^{1/2}$
$j = 1, 2,\ldots, n$; $k = 1, 2,\ldots\ldots, n$; $i = 1,2,\ldots\ldots,m$.

By similar arguments we obtain the relations for y_j as:

$$y_j = \frac{\frac{a}{\sqrt{a^2+b^2}}\sum_{\substack{k=1\\ \neq j}}^{n} \frac{E(v'_{jk})\,y_k}{\sqrt{(y_j - y_k)^2 + \varepsilon\frac{a^2}{a^2+b^2}}} + \frac{a^3}{a^2+b^2}\sum_{i=1}^{m}\frac{E(w_{ji})\hat{y}_i}{K_{ji}} - \frac{a^2b}{a^2+b^2}\sum_{i=1}^{m}\frac{E(w_{ji})\hat{x}_i}{K_{ji}} - \frac{abc}{a^2+b^2}\sum_{i=1}^{m}\frac{E(w_{ji})}{K_{ji}}}{\frac{a}{\sqrt{a^2+b^2}}\sum_{\substack{k=1\\ \neq j}}^{n}\frac{E(v'_{jk})}{\sqrt{(y_j - y_k)^2 + \varepsilon\frac{a^2}{a^2+b^2}}} + a\sum_{i=1}^{m}\frac{E(w_{ji})}{K_{ji}}} \quad \ldots (27A)$$

where $K_{ji} = \left[a^2(y_j - \hat{y}_i)^2 + (by_j + a\hat{x}_i + c)^2 + \varepsilon a^2\right]^{1/2}$
$j = 1, 2,\ldots, n$; $k = 1, 2,\ldots\ldots, n$; $i = 1, 2, \ldots\ldots, m$.

To solve the set of non-liner equations represented by (27) or (27A) we use the following iterative scheme:
The equation (27) can be written as

$x_j = H_j(x_1,x_2,\ldots, x_n)$, (j= 1, 2, ….. , n) ... (28)

Starting with initial solution $(x_1^{(0)}, x_2^{(0)},\ldots, x_n^{(0)})$ we form the successive approximate solutions $(x_1^{(1)}, x_2^{(1)},\ldots,x_n^{(1)}),(x_1^{(2)}, x_2^{(2)},\ldots, x_n^{(2)}),\ldots, (x_1^{(N)}, x_2^{(N)}, \ldots, x_n^{(N)})$ from the relations given by

$x_j^{(N+1)} = H_j (x_1^{(N)}, x_2^{(N)},\ldots, x_n^{(N)})$, … (29)

where the superscripts denote the iteration number. We take a feasible initial solution for the rapid convergence of the iterative scheme. The scheme will be convergent provided the following inequality holds:

$$\left|\frac{\partial H_j(t_{j1}, t_{j2}, \ldots, t_{jn})}{\partial x_k}\right| < \frac{M}{n} \text{ for all j and k,}$$

where $0 < M < 1$ and $t_{jk} = x_k^{(N)} + (\beta_k - x_k^{(N)})\theta_j$, $0 < \theta_j < 1$,
(k =1,2,…..,n) andβ_j are the solutions of (28). For the solution of (27A) in getting the values of y_j (j = 1,2,…..,n) we use the similar iterative scheme with a feasible initial solution. It may be pointed out that after obtaining the values of x_j (j = 1,2,….,n) by using the iterative scheme (29) one need not calculate the values of y_j(j = 1,2,…..,n) by using the iterative scheme as these can be found directly from the equation (24) with the help of determined values of x_j(j = 1,2,…..,n).

3. Numerical Example

Let us consider the example involving 2 new facilities and 2 existing facilities where the new facilities are supposed to be located in a triangular area given by $x_j + y_j - 2 \le 0$.Let the co-ordinates of the existing facilities be as follows:
$(\hat{x}_1, \hat{y}_1) = (2, 1)$ and $(\hat{x}_2, \hat{y}_2) = (1, 2)$.
Let the cost per unit distance among new facilities and between new and existing facilities and the corresponding probabilities be given as follows:

$$v_{(12)_1} = 3,\quad p_{(12)_1} = \frac{1}{3};\quad v_{(12)_2} = 3,\quad p_{(12)_2} = \frac{2}{3}; w_{(11)_1} = 4, q_{(11)_1} = \frac{3}{4};\quad w_{(11)_2} = 8,\quad q_{(11)_2} = \frac{1}{4};$$

$$w_{(12)_1} = 3,\; q_{(12)_1} = \frac{1}{3};\; w_{(12)_2} = 2,\quad q_{(12)_2} = \frac{1}{2};\; w_{(12)_3} = 6,\; q_{(12)_3} = \frac{1}{6};\; w_{(21)_1} = 6,\quad q_{(21)_1} = \frac{5}{6};$$

$$w_{(21)_2} = 12,\quad q_{(21)_2} = \frac{1}{6};\; w_{(22)_1} = 7,\quad q_{(22)_1} = \frac{6}{7}; w_{(22)_2} = 14,\quad q_{(22)_2} = \frac{1}{7}.$$

Let us take $\varepsilon = 1.2 \text{ X } 10^{-9}$.

3.1 Case – I $\lambda_j = 0$ (j = 1,2)

With these data we have the following iterative scheme:

$$x_1^{(N+1)} = \frac{\frac{5x_2^{(N)}}{\sqrt{(x_1^{(N)}-x_2^{(N)})^2+(y_1^{(N)}-y_2^{(N)})^2+\varepsilon}} + \frac{10}{\sqrt{(x_1^{(N)}-2)^2+(y_1^{(N)}-1)^2+\varepsilon}} + \frac{3}{\sqrt{(x_1^{(N)}-1)^2+(y_1^{(N)}-2)^2+\varepsilon}}}{\frac{5}{\sqrt{(x_1^{(N)}-x_2^{(N)})^2+(y_1^{(N)}-y_2^{(N)})^2+\varepsilon}} + \frac{5}{\sqrt{(x_1^{(N)}-2)^2+(y_1^{(N)}-1)^2+\varepsilon}} + \frac{3}{\sqrt{(x_1^{(N)}-1)^2+(y_1^{(N)}-2)^2+\varepsilon}}},$$

$$x_2^{(N+1)} = \frac{\frac{5x_1^{(N)}}{\sqrt{(x_2^{(N)}-x_1^{(N)})^2+(y_2^{(N)}-y_1^{(N)})^2+\varepsilon}} + \frac{14}{\sqrt{(x_2^{(N)}-2)^2+(y_2^{(N)}-1)^2+\varepsilon}} + \frac{8}{\sqrt{(x_2^{(N)}-1)^2+(y_2^{(N)}-2)^2+\varepsilon}}}{\frac{5}{\sqrt{(x_2^{(N)}-x_1^{(N)})^2+(y_2^{(N)}-y_1^{(N)})^2+\varepsilon}} + \frac{7}{\sqrt{(x_2^{(N)}-2)^2+(y_2^{(N)}-1)^2+\varepsilon}} + \frac{8}{\sqrt{(x_2^{(N)}-1)^2+(y_2^{(N)}-2)^2+\varepsilon}}},$$

$$y_1^{(N+1)} = \frac{\frac{5y_2^{(N)}}{\sqrt{(x_1^{(N)}-x_2^{(N)})^2+(y_1^{(N)}-y_2^{(N)})^2+\varepsilon}} + \frac{5}{\sqrt{(x_1^{(N)}-2)^2+(y_1^{(N)}-1)^2+\varepsilon}} + \frac{6}{\sqrt{(x_1^{(N)}-1)^2+(y_1^{(N)}-2)^2+\varepsilon}}}{\frac{5}{\sqrt{(x_1^{(N)}-x_2^{(N)})^2+(y_1^{(N)}-y_2^{(N)})^2+\varepsilon}} + \frac{5}{\sqrt{(x_1^{(N)}-2)^2+(y_1^{(N)}-1)^2+\varepsilon}} + \frac{3}{\sqrt{(x_1^{(N)}-1)^2+(y_1^{(N)}-2)^2+\varepsilon}}}$$

and

$$y_2^{(N+1)} = \frac{\frac{5y_1^{(N)}}{\sqrt{(x_2^{(N)}-x_1^{(N)})^2+(y_2^{(N)}-y_1^{(N)})^2+\varepsilon}} + \frac{7}{\sqrt{(x_2^{(N)}-2)^2+(y_2^{(N)}-1)^2+\varepsilon}} + \frac{16}{\sqrt{(x_2^{(N)}-1)^2+(y_2^{(N)}-2)^2+\varepsilon}}}{\frac{5}{\sqrt{(x_2^{(N)}-x_1^{(N)})^2+(y_2^{(N)}-y_1^{(N)})^2+\varepsilon}} + \frac{7}{\sqrt{(x_2^{(N)}-2)^2+(y_2^{(N)}-1)^2+\varepsilon}} + \frac{8}{\sqrt{(x_2^{(N)}-1)^2+(y_2^{(N)}-2)^2+\varepsilon}}}$$

Starting with a feasible initial solution $(x_1^{(0)}, x_2^{(0)}, y_1^{(0)}, y_2^{(0)})$ as (1.00,1.10,0.85,1.15) we obtain the successive approximate solutions and get the values of x_1, x_2, y_1, y_2 as:
$x_1 = 1.27$, $x_2 = 1.31$, $y_1 = 1.22$, $y_2 = 1.26$.Thus the locations of the two new facilities are given by (1.27, 1.22) and (1.31, 1.26). But this solution does not satisfy the given constraint $x_j + y_j - 2 \le 0$and hence this is not a feasible solution. Therefore we have to consider the case when $\lambda_j \neq 0$ (j = 1,2) which will give the optimal location necessarily on the boundary.

3.2 Case – II $\lambda_j \neq 0$ (j = 1,2)

With the given data we have the following iterative scheme:

$$x_1^{(N+1)} = \frac{\frac{1}{\sqrt{2}}\frac{5x_2^{(N)}}{\sqrt{(x_1^{(N)}-x_2^{(N)})^2+\frac{\varepsilon}{2}}}+\frac{15}{2}\frac{1}{T_{11}^{(N)}}+\frac{3}{2}\frac{1}{T_{12}^{(N)}}}{\frac{1}{\sqrt{2}}\frac{5}{\sqrt{(x_1^{(N)}-x_2^{(N)})^2+\frac{\varepsilon}{2}}}+\frac{5}{T_{11}^{(N)}}+\frac{3}{T_{12}^{(N)}}} \text{ and}$$

$$x_2^{(N+1)} = \frac{\frac{1}{\sqrt{2}}\frac{5x_1^{(N)}}{\sqrt{(x_2^{(N)}-x_1^{(N)})^2+\frac{\varepsilon}{2}}}+\frac{21}{2}\frac{1}{T_{21}^{(N)}}+\frac{4}{T_{22}^{(N)}}}{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{(x_2^{(N)}-x_1^{(N)})^2+\frac{\varepsilon}{2}}}+\frac{7}{T_{21}^{(N)}}+\frac{8}{T_{22}^{(N)}}},$$

where $T_{11}^{(N)} = \left[\left(x_1^{(N)}-2\right)^2+\left(x_1^{(N)}-1\right)^2+\varepsilon\right]^{1/2}$,

$$T_{12}^{(N)} = \left[\left(x_1^{(N)}-1\right)^2+\left(x_1^{(N)}\right)^2+\varepsilon\right]^{1/2},$$

$$T_{21}^{(N)} = \left[\left(x_2^{(N)}-2\right)^2+\left(x_2^{(N)}-1\right)^2+\varepsilon\right]^{1/2} \text{ and}$$

$$T_{22}^{(N)} = \left[\left(x_2^{(N)}-1\right)^2+\left(x_2^{(N)}\right)^2+\varepsilon\right]^{1/2}.$$

Starting with a feasible initial solution $(x_1^{(0)}, x_2^{(0)})$ as (1.0, 1.5) we get the successive approximate solutions and obtain the values of x_1 and x_2 as x_1 = 1.20 and x_2 = 1.70. We substitute these values of x_1 andx_2 in the equation (24) when j = 1 and 2 respectively and we get y_1 = 0.80 and y_2 = 0.30. Thus the locations of the two new facilities are (1.2, 0.8) and (1.7,0.3).

4. Location of New Facilities in Semi-open Rectangular Areas

We consider the problem of locating new facilities with respect to multiple existing facilities in a semi-open rectangular area which may be given either by

$ax_j + c \leq 0$ (a> 0, c < 0) or $by_j + c \leq 0$ (b > 0, c < 0). …(30)

Clearly this type of area is a special case of the area given by (2) when either b=0 or a = 0 respectively. The equations $ax_j + c = 0$ or $by_j + c = 0$ represent straight lines parallel either to y-axis or x-axis respectively. Thus the only difference is that the area is now open whereas in the case considered in section-2, the area is closed. However, the solution procedure described in section-2 works equally well for these cases also. The unconstrained optimum corresponding to the case $\lambda_j = 0$ will be the same as given under thesection-2.1. If this solution does not satisfy (30), we have to consider the case when $\lambda_j \neq 0$ (j = 1,2,….., n). Let us consider any one of the two areas, say,

$ax_j + c \leq 0$ (a> 0, c < 0). … (31)

When $\lambda_j \neq 0$ (j = 1,2,….., n), we have by one of the Kuhn-Tucker conditions $ax_j + c = 0$ which makes all x_j fixed and is given by $x_j = -\frac{c}{a}$, (j = 1,2,….., n). Now in finding the y-co-ordinates of the new facilities we get the expressions for y_j (j = 1,2,….., n) simply by putting b = 0 in the equation (27A). Thus the expressions for y_j are given by:

$$y_j = \frac{\sum_{\substack{k=1\\ \neq j}}^{n}\frac{E(v'_{jk})\,y_k}{\sqrt{(y_j-y_k)^2+\varepsilon}}+a\sum_{i=1}^{m}\frac{E(w_{ji})\,\hat{y}_i}{K_{ji}}}{\sum_{\substack{k=1\\ \neq j}}^{n}\frac{E(v'_{jk})}{\sqrt{(y_j-y_k)^2+\varepsilon}}+a\sum_{i=1}^{m}\frac{E(w_{ji})}{K_{ji}}}, \quad \text{… (32)}$$

where $K_{ji} = \left[a^2\left(y_j-\hat{y}_i\right)^2+(a\hat{x}_i+c)^2+\varepsilon a^2\right]^{1/2}$.

For the solution of (32) we use the iterative scheme described under section-2.2. If the area is given by

$by_j + c \leq 0$ (b> 0, c < 0), …(33)

in a similar way the y-co-ordinates of the new locations are given by $y_j = -\frac{c}{b}$, (j = 1,2,….., n) and the expressions for x_j (j = 1,2,….., n) can be found simply by putting a = 0 in the equation (27) as:

$$x_j = \frac{\sum_{\substack{k=1\\ \neq j}}^{n}\frac{E(v'_{jk})x_k}{\sqrt{(x_j-x_k)^2+\varepsilon}}+b\sum_{i=1}^{m}\frac{E(w_{ji})\,\hat{x}_i}{T_{ji}}}{\sum_{\substack{k=1\\ \neq j}}^{n}\frac{E(v'_{jk})}{\sqrt{(x_j-x_k)^2+\varepsilon}}+b\sum_{i=1}^{m}\frac{E(w_{ji})}{T_{ji}}}, \quad \text{… (34)}$$

where$T_{ji} = \left[b^2\left(x_j-\hat{x}_i\right)^2+(b\hat{y}_i+c)^2+\varepsilon b^2\right]^{1/2}$

For the solution of (34) we use the iterative scheme describedunder section-2.2. It may be mentioned that for solving the problems where the new facilities have to be located on a straight line only the algorithm developed under section-2.2 can be utilized.

5. Conclusion

A point by point writing overview uncovers that a little consideration has been paid to multifacility location issue including area restriction despite the fact that; maybe every location issue is intrinsically bound by territory requirement or something to that affect or the other. It might likewise be specified that the work containing stochastic contemplations has additionally gotten less consideration despite the fact that practical nature of the issues requests probabilistic examinations. This has inspired to deal with the area limitation part of stochastic multifacility location issue in the present study. Multifacility location issue under triangular territory confinement with probabilistic methodology for the weights considered in the objective function and the Euclidean distance between the locations has been talked about.

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