On Strong Split Middle Domination of a Graph

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Abstract:-The middle graph $M(G)$ of graph G is obtained by inserting a vertex x_i in the "middle" of each edge e_i, $1 \leq i \leq |E(G)|$, and adding the edge x_ix_j for $1 \leq i \leq j \leq |E(G)|$ if and only if e_i and e_j have a common vertex. A dominating set D of graph G is said to be a strong split dominating set of G if $\langle V(G) - D\rangle$ is totally disconnected with at least two vertices. Strong split domination number is the minimum cardinality taken over all strong split dominating sets of G.

In this paper we initiate the study of strong split middle domination of a graph. The strong split middle domination number of a graph G, denoted as $\gamma_{ssm}(G)$ is the minimum cardinality of strong split dominating set of $M(G)$. In this paper many bounds on $\gamma_{ssm}(G)$ are obtained in terms of other domination parameters and elements of graph G. Also some equalities for $\gamma_{ssm}(G)$ are established.

Keywords: *Domination Number, Strong Split Domination Number, Middle Graph.*

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I. Introduction

All the graphs considered here are simple, finite, nontrivial, undirected and connected. The vertex set and edge set are $V[G]$ and $E(G)$ respectively with $|V(G)|=p$ and $|E(G)=q$. Terms not defined here and used in the sense of Harary [2].

The degree, neighborhood and closed neighborhood of a vertex v in a graph G are denoted as $\deg(v)$, $N(v)$ and $N[v] = N(v) \cup v$ respectively. For a set $S \subseteq V$, the graph induced by S is denoted as $\langle S\rangle$.

Δ(G) ($\Delta'(G)$) denotes the maximum degree of a vertex (edge) in G.

A set $H \subset V(E)$ is said to be a vertex/edge cover if it covers all the edges /vertices of G. The minimum cardinality over all the vertex/edge covers is called vertex/edge covering number and is denoted by $\alpha_0(G)/\alpha_1(G)$. A set $H_1 \subset V/(E)$ in a graph is said to be independent set if no two vertices/edges are adjacent. The vertex/edge independence number $\beta_0/\beta_1(G)$ is the maximum cardinality of an independent set of vertices/edges.

A line graph $L(G)$ is a graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A dominating set S is called a connected dominating set if the induced subgraph $\langle S\rangle$ is connected. The minimum cardinality taken over all minimal connected dominating sets is called connected domination number and is denoted by $\gamma_c(G)$.

A set $F \subset E(G)$ is said to be edge dominating set of G if every edge in $\langle E(G) - F\rangle$ is adjacent to at least one edge in F. The minimum cardinality taken over all edge dominating sets of G, denoted as $\gamma'(G)$ is called edge domination number of G.

Given two adjacent vertices u and v of G. We say u strongly dominates v if $\deg u \geq \deg v$. A set $D \subseteq V(G)$ is a strongly dominating set if every vertex in $\langle V - D\rangle$ is strongly dominated by at least one vertex in D. The strong domination is introduced by Sampathkumar et. al [11]. Strong domination number is the cardinality of minimal strong dominating sets of G and is denoted as $\gamma_{st}(G)$.

A set S of elements of G is an entire dominating set of G, if every element not in S is either adjacent or incident to at least one element in S. The entire domination number $\gamma_{en}(G)$ is the cardinality of a smallest entire dominating set. This concept was introduced by Kulli in [3].

A dominating set D of a graph G is a split dominating set if the induced subgraph $\langle V - D\rangle$ is disconnected. The split domination number $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set. The split domination number is introduced and discussed in [4].

A dominating set of $D \subseteq V$ is called strong split dominating set of G, if $\langle V - D\rangle$ is totally disconnected with at least two vertices. The minimum cardinality taken over all the strong split dominating sets of G is called as strong split domination number of G. It is introduced in [6].

The middle graph $M(G)$ of a graph G is obtained by inserting a vertex x_i in the middle of each edge e_i, $1 \leq i \leq |E(G)|$, and adding the edge $x_i x_j$ for $1 \leq i \leq j \leq |E(G)|$ if and only if e_i and e_j are adjacent in G.

The regular number of middle graph M(G) of G is the minimum number of subsets into which the edge set of M(G) should be partitioned so that the subgraph induced by each subset is regular and is denoted by r_m G.This concept is discussed in[10].

In this paper we introduce a new variation of domination parameter as strong split middle domination of a graph G. The strong split domination number of a middle graph of graph G is referred here as strong split middle domination number of G and is denoted as $\gamma_{ssm}(G)$.

Strong Split Line domination number ($\gamma_{ssl}(G)$), strong split block cut vertex domination number ($\gamma_{ssbc}(G)$) and strong split lict domination number ($\gamma_{ssn}(G)$) are introduced in [9], [8] and [7] respectively.

In [1], Allan and Laskar have shown middle graphs with equal independence domination number and domination number. In this paper we find equality for strong split domination number of $M(G)$ in terms of elements of G and connected domination number of $M(G)$. Also we obtain many lower bounds and upper bounds in terms of $\gamma_{ssl}(G)$ $\gamma_{ssbc}(G)$, $\gamma_{ssn}(G)$. We obtained some more inequalities in terms of strong split domination number of semitotal block graph of G.

II. Prerequisites

Theorem A [6]. If G is a graph without isolated vertices and $p \geq 3$, then $\gamma_{ss}(G) = \alpha(G)$.

Theorem B [5]. For any graph G

$$p - \frac{2}{3} q \leq \gamma_{cot}(G).$$

Theorem C [10]. For any nontrivial tree T, with n-cut vertices with same degree and $n \geq 2$, $r_m(T) = 3$.

III. RESULTS

Theorem 1. Let $A = \{v'_1, v'_2 \ldots\ldots\ldots\ldots v'_n\}$ be the set of vertices which divide each edge of G. Then A is the γ_{ssm}-set of G.

Proof. Let $A = \{v'_1, v'_2 \ldots\ldots\ldots\ldots v'_q\}$ be the set of vertices which divides each edge of G. Then $V[M(G)] = V(G) \cup A$. Clearly each $v_i \in A$ dominates two vertices $u, v \in V(G)$ and some $v_j \in A$, which subdivides the edge e_j incident to u or v in G. Then A is a dominating set, also $\langle V(M(G)) - A\rangle$ is a totally disconnected graph. Thus A is a strong split middle dominating set of G. Now without loss of generality we consider a vertex $u' \in A$, for which $u, v \in V[G]$ and $u, v \in N$ (u′). Now $(A - u')$ is a dominating set of G. But it is clear that there exists exactly two edges uu' and $u'v$ in the edge set of $\langle V[M(G)] - (A - u')\rangle$, thus by the definition $(A - u')$ is not a γ_{ss} – set of $M(G)$. Further consider $D' = (A - u') \cup (u, v)$, clearly $N[D'] = V[M(G)]$. So D' is a dominating set of $M(G)$, such that $\langle V[M(G)] - D'\rangle$ is totally disconnected, Hence D' is a strong spilt dominating set. But $|D'| = |(A - u') \cup \{u, v\}| > |A|$, hence A is the minimal strong split middle dominating set of G. □

Theorem 2. For any connected (p, q) graph G. $\gamma_{ssm}(G) = q$.
Proof. From Theorem 1, $\gamma_{ssm}[G] = |A|$ and $|A| = q$. Hence the desired result. □

In the following theorem an equality between strong split middle domination number and connected middle domination number, for a tree is established.

Theorem 3. For any tree T, $\gamma_{ssm}(T) = \gamma_c[M(T)]$
Proof. For any (p, q) tree T. Let $A = \{v_1', v_2', \ldots\ldots\ldots\ldots v_q'\}$, each v_i, $1 \leq i \leq q$ divides the edges e_i of G, thus $V[M(T)] = V(T) \cup A$. Now without loss of generality consider two edges e_i, $e_j \in E(G)$, if e_i, e_j are adjacent G, then v_1', v_j' are adjacent in $M(G)$, clearly $N[A] = V[M(T)]$. Further suppose $\exists\, v_k' \in A$ such that $V[M(T)] - \{A - (v_k')\}$ gives atleast one edge in $\langle V[M(T)] - \{A - (v_k'J\}$. Hence A is minimal γ_{ssm} – set of G. As T is connected, there exist at least one path between every pair of vertices of A, then $\langle A\rangle$ is connected. Hence $\gamma_{ssm}(T) = \gamma_c[M(T)]$. □

Theorem 4. For any connected (p, q) graph G. $\gamma_{ssm}[G] \geq p - 1$, equality holds for a tree.
Proof. Let G be a (p, q) tree, then from theorem 2. $\gamma_{ssm}[G] = q$. Hence $\gamma_{ssm}[G] = p - 1$. Further suppose G is not a tree, then there exists a cycle in G, clearly $q > p - 1$, Again from Theorem 2 $\gamma_{ssm}[G] > p - 1$. □

Theorem 5. For any connected (p, q) graph G. $\gamma[M(G)] \leq \gamma_{ssm}(G)$, equality holds for $K_{1,m}$, m ≥ 1.
Proof. Suppose $D = \{v_1, v_2, \ldots\ldots\ldots v_n\} \subseteq V[M(G)]$ be the minimal set of vertices such that $N[D] = V[M(G)]$, then D is a minimal dominating set of $M(G)$. Further let $A = \{v_1', v_2', \ldots\ldots\ldots v_q'\}$ be the set of vertices subdividing the edges of G in $M(G)$. Then from Theorem 1, $\langle V(M(G)) - A\rangle$ is totally disconnected with at least two vertices, and $|A| = q$. Clearly $|D| \leq |A|$, resulting in to $\gamma[M(G)] \leq \gamma_{ssm}(G)$.
For the equality
Suppose $G = K_{1,m}$, m ≥ 1. Then $V[M(K_{1,m})] = V[K_{1,m}] \cup A$. Such that $\langle V[M(K_{1,m})] - A\rangle$ is totally disconnected with at least two vertices, and A is minimal dominating set of $M(K_{1,m})$. Hence $|A| = \gamma_{ssm}[K_{1,m}] = \gamma[M(K_{1,m})]$. □

The following theorem gives a lower bound for $\gamma_{ssm}(G)$ in terms of edge domination number and maximum degree of the graph.

Theorem 6. For any connected (p, q) graph G

$$\gamma'(G) + \Delta'(G) \leq \gamma_{ssm}(G).$$

Proof. Let F = $\{e_1', e_2', \ldots\ldots\ldots e_m'\}$ be a minimal set of edges, such that $N[F] = E(G)$, then from definition of edge dominating set F is a γ'-set of G. Suppose $e_i \in E(G)$ is the maximum edge degree in G. Let F′ = $\{e_1, e_2 \ldots . e_n\}$ be the set of edges such that $N(F') \subset$

F and $|F'| = \Delta'(G)$. Thus $|F| \leq |E(G) - \Delta'(G)|$, Further let $S = \{v_1', v_2' \ldots\ldots\ldots v_q'\}$ be the set of vertices dividing each edge of G. then $V[M(G)] = V(G) \cup S$ and $N[S] = V[M(G)]$. It is clear that $\langle V[M(G)] - S \rangle$ is totally disconnected. Then S is a γ_{ss} - set of $M(G)$. By Theorem 1 , S is a minimal γ_{ss} – set of M(G). $|S| = |E(G)| = \gamma_{ss}[M(G)]$. Therefore it follows that $|F| \leq |S| - \Delta'(G)$, resulting in $\gamma'(G) + \Delta'(G) \leq \gamma_{ssm}(G)$. □

Theorem 7. For any connected (p, q) graph G, $\gamma'(G) + \beta_1 \leq \gamma_{ssm}(G)$.

Proof. Let $F_1 = \{e_1, e_2, \ldots\ldots\ldots e_q\} \leq E(G)$ be the maximal set of edges with $N(e_i) \cap N(e_j) = e_k$ for every $e_i, e_j \in F$, $1 \leq j \leq n$ and $e_k \in E(G) - F$, clearly F forms a maximal independent edge set in G. Hence $|F| = \beta_1(G)$. Suppose $F_2 \subseteq E(G)$ be a minimal set of edges such that each edge in $E(G) - F_2$ is adjacent to at least one edge in F_2. Then F_2 forms an edge dominating set of G. Clearly $F_1 \cup F_2 \subseteq E(G)$ and Theorem 2 it follows that $\gamma'(G) + \beta_1 \leq \gamma_{ssm}(G)$. □

Following theorem relates entire dominating number of a graph with $\gamma_{ssm}(G)$

Theorem 8. For any connected graph G. $p - \gamma_{en}(G) = \leq \gamma_{ssm}[M(G)]$ equality is attained if any only if G is a star.

Proof. Let $S = D_1 \cup F_1$, be the minimum entire dominating set of G, where $D_1 \subseteq V[G]$ and $F_1 \subseteq E(G)$. Then

$$\begin{aligned} P - |S| &= |V(G) - S| \\ &\leq |V(G)| - 1 \\ &\leq p - (p - q) \\ &\leq q \end{aligned}$$

Since $\gamma_{ssm}(G) = q$, we have $p - \gamma_{en}(G) \leq \gamma_{ssm}(G)$.

Suppose $p - \gamma_{en}(G) = \gamma_{ssm}(G)$, then $p - \gamma_{en}(G) \geq 1$.

From the above inequalities we have $p - q = 1$ gives $p - \gamma_{ssm}[G] = 1 = \gamma_{en}(G)$, which shows G is a star.

Converse is obvious. □

In the following theorem we establish both lower bound and upper bound for our concept.

Theorem 9. For any graph G,

$$p - 1 \leq \gamma_{ssm}(G) \leq \frac{p(p-1)}{2}$$

Proof. For any minimal connected graph G, the number of edges is $p - 1$, similarly the maximum number of edges in a graph G is $\frac{p(p-1)}{2}$. From Theorem 2, both lower and upper bounds are attained. □

Theorem 10. For any connected (p, q) graph G,

$$\gamma_{ssbc}(G) \leq \gamma_{ssm}(G)$$

Equality holds for a tree with $p \geq 3$ vertices.

Proof. First we prove the equality for a tree. Let $B = \{b_1, b_2, \ldots\ldots\ldots b_n\}$ be the set of vertices corresponding to the blocks of a tree T, and let $C = \{C_1, C_2, \ldots\ldots C_m\}$ be the set of cut vertices of tree T. Then $V[BC(G)] = B \cup C$, clearly for each $C_i \in C$, $\deg(b_j) \geq \deg(C_i)$, where for each b_j, $1 \leq j \leq n$; $b_j \in N(C_i)$ in $BC(G)$. Also each block vertex in $N(C_i)$ is adjacent to at least one block vertex in $N(C_i)$, thus the set of block vertices B is such that every vertex in $[V[BC(G)] - B]$ is adjacent to at least two vertices of B. Thus B is a dominating set of G. Further B is a minimal set of vertices for which $\langle V[BC(G)] - B \rangle$ is totally disconnected, thus B forms a strong split block cut vertex dominating set of G. Thus $|B| = \gamma_{ssbc}(G)$. If $S = \{v_1', v_2', \ldots\ldots\ldots v_q'\}$ be the set of vertices, subdividing the set of edges of T. Then by Theorem 1, S is the γ_{ssm}-set of T. Clearly $|S| = \gamma_{ssm}[T]$, and $|B| = |S|$ gives $\gamma_{ssbc}[T] = \gamma_{ssm}[T]$.

Suppose G is not a tree. Then there exists an edge joining any two non adjacent vertices of T. Hence $E[M(G)] > |B|$, again by Theorem 2. $|S| > |B|$ which gives $\gamma_{ssm}(G) > \gamma_{ssbc}(G)$. Thus the desired result $\gamma_{ssbc}(G) \leq \gamma_{ssm}(G)$. □

The following theorem shows that $\gamma_{ss}(G)$ is an upper bound to $\gamma_s(G)$ and $\gamma_{ss}(G)$.

Theorem 11. For any graph G

$$\gamma_s(G) \leq \gamma_{ss}(G) \leq \gamma_{ssm}[G]$$

Proof. First we prove the upper bound. Let $D = \{v_1, v_2, \ldots\ldots\ldots v_m\}$ be the minimal set of vertices such that $N[D] = V[G]$ and $\langle V[G] - D \rangle$ is totally disconnected with at least two vertices, then D forms a γ_{ss} – set of G. From corollary [A], $|D| = \alpha_0(G)$ implies $\gamma_{ss}(G) < p$. Since for any graph G, $q \geq p - 1$, then $\gamma_{ss}(G) \leq q$. From Theorem 2 $\gamma_{ss}(G) \leq \gamma_{ssm}(G)$.

To prove the lower bound, consider a minimum set of vertices D_1, such that $N[D_1] = V(G)$ and $\langle V - D_1 \rangle$ is disconnected, it follows that D_1 is a split dominating set of G. Also it is clear that $D_1 \subseteq D$, resulting into $|D_1| \leq |D|$, i.e. $\gamma_s(G) \leq \gamma_{ss}(G)$. □

The Theorem 12 and 13 show that $\gamma_{ssm}(G)$ is an upper bound to strong domination and strong middle domination number of G.

Theorem 12. For any connected (p, q) graph G,

$$\gamma_{st}(G) \leq \gamma_{ssm}(G)$$

Proof. For the graph $M(G)$, $V[M(G)] = V(G) \cup S$, where S is the set of vertices subdividing each edge of G. Clearly $|S| = |E(G)|$, from Theorem 2, $|S| = \gamma_{ssm}(G)$. Also we consider a set of vertices $D = \{v_1, v_2, \ldots\ldots v_n\}$ such that $N[D] = V(G)$ and for each $v_i \in D$. $\exists$ a $u_i \in N(D)$ and $\deg v_i \geq \deg u_i$, thus D is a strong dominating set of G and clearly $\gamma_{st}(G) < p$. From Theorem 4, $\gamma_{ssm}(G) \geq p - 1$, consequently $\gamma_{ssm}(G) \geq (G)$. □

Theorem 13. For any connected (p, q) graph G $\gamma_{sm}(G) \leq \gamma_{ssm}(G)$

Proof. Let $S = \{v_1', v_2', \ldots\ldots v_n'\}$ be the set of vertices dividing each edge of G, then $|S| = |E(G)|$, S is the minimum set such that $S \subseteq V[M(G)]$ and $\langle V[M(G)] - S\rangle$ is a null graph. Then $|S| = \gamma_{ssm}(G)$.

Further each v_i' divides an edge e_i, for which $\deg e_i = m$, then $\deg v_i' = m + 2$, also $\deg v_i' \geq \deg u_i$, $u_i \in V[M(G)] - S']$, where $S' \subseteq S$, then S' is the strong dominating set of $M[G]$. And since $S' \subseteq S$, it follows $\gamma_{sm}(G) \leq \gamma_{ssm}(G)$. □

The following theorem gives equality between strong split middle domination and strong split lict domination number of a graph.

Theorem 14. For a tree T with at least two cut vertices $\gamma_{ssm}(T) = \gamma_{ssn}(T)$.

Proof. For a tree, suppose $E(T) = \{e_1, e_2, \ldots\ldots e_q\}$ and $C = \{C_1, C_2, \ldots\ldots C_k\}$ be the set of edges and cut vertices respectively. In $n(G)$, $V[n(G)] \subset A \cup C$ where $A = \{v_1, v_2, \ldots\ldots v_q\}$ is the set of vertices corresponding to each element of E. Since each block in E is complete, the each $v_i \in \langle V[n(G)] - A\rangle$ is adjacent to at least one $v_k \in A$ and for each $v_i \in \langle V[n(G)] - A\rangle$, $\deg v_i = 0$. Hence A is a minimal γ_{ssn}-set of T. For any nontrivial tree each block is an edge. By Theorem 2, $E(T)$ forms a γ_{ssm} -set, and also $|A| = q$, which gives $\gamma_{ssm}(T) = \gamma_{ssn}(T)$. □

In the following theorem we relate γ_{ssm}-number of a graph with regular number $r_m(G)$ of a middle graph of a graph.

Theorem 15. For any nontrivial tree T, with $p \geq 3$ vertices, $\gamma_{ssm}(G) \geq r_m(G)$

Proof. For any nontrivial tree T, with $p = 2$ vertices $\gamma_{ssm}(G) = 1$ and $r_m(G) = 2$. Further consider a nontrivial tree T with $p \geq 3$vertices and by Theorem C, $r_m(T) = 3$.Since by Theorem 2, $\gamma_{ssm}(T) \geq 3$. Hence $\gamma_{ssm}(T) \geq r_m(T)$. □

Theorem 16. For any connected (p, q) graph G. $\gamma_{ssl}[G] \leq \gamma_{ssm}(G)$.

Proof. Let $V_1' = \{v_1', v_2', \ldots\ldots v_q'\}$ be the set of vertices corresponding to the edges of G. Then $V[L(G)] = V_1'$ and $V[M(G)] = E \cup V_1'$. Let D is a minimal set of vertices of $L(G)$ such that $N[D] = V_1'$ and $\langle V_1 - D\rangle$ is totally disconnected with at least two vertices, then D is a strong split dominating set of $L(G)$. Thus $|D| = \gamma_{ssl}(G)$. Further consider a set $D' \subseteq V[M(G)]$ such that D' is a minimal dominating set satisfying the condition $\langle V[M(G)] - D'\rangle$ is totally disconnected with at least two vertices. Since $V[n(G)] > V[L(G)]$, then by Theorem 15, $|D| \leq |D'|$ which gives $\gamma_{ssl}(G) \leq \gamma_{ssm}(G)$. □

Theorem 17. For any graph G $p - \gamma_{cot}(G) \leq \frac{2}{3}\gamma_{ssm}(G) - p$.

Proof. From Theorem B, $p - \frac{2}{3}q \leq \gamma_{cot}(G)$. And using Theorem 2, the result follows. □

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