Fixed point theorem in fuzzy metric space for owc maps satisfying integral type inequality

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Abstract

This paper introduces the fixed point in complete fuzzy metric space, and how to find a common fixed point between occasionally weakly compatible mappings

Keywords: Fuzzy metric space, occasionally weakly compatible mappings, common fixed point.

1. Introduction

Many mathematicians have studied common fixed points in fuzzy metric space under several conditions. In 1986 Jungck introduced the notion of compatible maps for a pair of self maps. Several papers have come up involving compatible maps in proving the existence of common fixed points both in the classical and fuzzy metric space. Recently Chouhan and Badshah (2010) established fixed points theorem in fuzzy metric spaces for weakly compatible maps. This paper introduces the common fixed point between occasionally weakly compatible mappings.

2. Preliminaries

Definition(2.1):[3] A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a t-norm if $*$ is satisfies the following condition:

(i) $*$ is commutative and associative;

(ii) $a * 1 = a \; for \; all \; a \in$ I;

(iii) $a * b \leq c * d \; whenever \; a \leq c and \; b \leq d \; for \; all \; a, b, c, d \in$ [0,1].

Definition (2.2): [3] Let X be a non-empty set, $*$ be a continuous t-norm on I= [0, 1]. A function $M : X \times X \times (0,\infty) \to [0,1]$ is called a fuzzy metric function on X if it satisfies the following axioms: $for\ all\ x,y,z \in X\ and\ for\ all\ t,s > 0$

(1) $M\ (x,y,t) > 0$;
(2) $M\ (x,y,t) = 1 \leftrightarrow x = y$;
(3) $M\ (x,y,t) = M\ (y,x,t)$;
(4) $M\ (x,y,t+s) \geq M\ (x,z,t) * M\ (z,y,s)$;
(5)$M\ (x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Note that $M\ (x,y,t)$ can be thought of as the degree of nearness between $x\ and\ y$ with respect to t. And $(X,M,*)$ is called a fuzzy metric space.

Definition (2.3): [1] Let X be a set, f,g self maps of X. A point x in X is called a coincidence point of f and g iff $f_x = g_x$. We shall call $w = f_x = g_x$ a point of coincidence of f and g.

Definition (2.4): [1] A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

The concept occasionally weakly compatible is introduced by M. Al-Thagafi and Naseer Shahzad [4]. It is stated as follows.

Definition (2.5): [1] Two self maps f,g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of $f\ and\ g$ at which $f\ and\ g$ commute.

A occasionally weakly is weakly compatible but converse is not true.

Example (2.6): Let R be the usual metric space. Define $S,T : R \to R$ by $S_x = 9x$ and $T_x = x^3$ for all $x \in R$. Then $S_x = T_x$ for $x = 0$, 3 but $ST_0 = TS_0$, and $ST_3 \neq TS_3$. S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma (2.7): [1] Let X be a set, f,g owc self maps of X. If f and g have a unique point of coincidence, $w = f_x = g_x$, then w is the unique common fixed point of f and g.

Lemma (2.8): [1] Let $(X,M,*)$ be a fuzzy metric space. If there exists q ∈ (0, 1) such that $M\ (x,y,qt) \geq M\ (x,y,t)$for all $x,y \in X$ and $t > 0$, then $x = y$.

Definition (2.9): [5] Let $(X,M,*)$ be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X is said to be fuzzy convergent to x in X if for each $\varepsilon \in (0,1)$and each $t > 0$, there exist $n_0 \in Z^+$ such that $M\ (x_n,x,t) > 1-\varepsilon$ for all $n \geq n_0$ (or equivalent $\lim_{n\to\infty} M\ (x_n,x,t) = 1$).

(b) A sequence $\{x_n\}$ in X is said to be fuzzy Cauchy sequence if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exist $n_0 \in Z^+$ such that $M\ (x_n,x_m,t) > 1-\varepsilon$ for all $n\ ,m \geq n_0$ (or equivalent $\lim_{n,m\to\infty} M\ (x_n,x_m t) = 1$).

(c) A fuzzy metric space in which every fuzzy Cauchy sequence is fuzzy convergent is said to be complete.

2. Main Results

Lemma (3.1): Let $(X, M,*)$ be a fuzzy metric space. If there exists q ∈ (0, 1) such that $\int_0^{M(x,y,qt)} \phi(t)dt \geq \int_0^{M(x,y,t)} \phi(t)dt$ such that $x, y \in X$ and $t > 0$, $\phi: R^+ \to R$ is Lebesgue-integrable and increasing mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for all $\varepsilon > 0$, then $x = y$.

Theorem (3.2): Let $(X, M,*)$ be a complete fuzzy metric space and let P, Q, S and T are self-mappings of X. Let the pairs {P, S} and {Q, T} be owc. If there exists q ∈(0, 1) such that

$$\int_0^{M(P_x,Q_y,qt)} \phi(t)dt \geq$$
$$\int_0^{min\{M(S_x,T_y,t),[M(S_x,P_x,t).M(Q_y,T_y,t)],M(P_x,T_y,t),M(Q_y,S_x,t)\}} \phi(t)dt \ldots. (1)$$

For all $x, y \in X$ and $t > 0$, $\phi: R^+ \to R$ is Lebesgue-integrable and increasing mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for all $\varepsilon > 0$.Then P, Q, S and T have a unique common fixed point.

Proof: Since pairs of mappings {P, S} and {Q, T} are owc, there exists two points $x, y \in X$ such that $P_x = Q_y$

$$\int_0^{M(P_x,Q_y,qt)} \phi(t)dt \geq \int_0^{min\{M(S_x,T_y,t),[M(S_x,P_x,t).M(Q_y,T_y,t)],M(P_x,T_y,t),M(Q_y,S_x,t)\}} \phi(t)dt$$

$$=\int_0^{min\{M(P_x,Q_y,t),[1.1],M(P_x,Q_y,t),M(P_x,Q_y,t)\}} \phi(t)dt = \int_0^{M(P_x,Q_y,t)} \phi(t)dt$$

$\Rightarrow P_x = Q_y$ then $P_x = S_x = Q_y = T_y$

Suppose that z such that $P_z = S_z$ then by (1) we have $P_z = Q_z \Rightarrow P_z = S_z = Q_y = T_y$ so $P_x = P_z$ and $w = P_x = S_x$ is the unique point of P and S. Similarly there is $z \in X$ such that $z = Q_z = T_z$.

Assume $z \neq w$ ⇒by (1) we have

$$\int_0^{M(w,z,qt)} \phi(t)dt = \int_0^{M(P_x,Q_z,qt)} \phi(t)dt \geq$$
$$\int_0^{min\{M(S_x,T_z,t),[M(S_x,P_x,t).M(Q_z,T_z,t)],M(P_x,T_z,t),M(Q_z,S_x,t)\}} \phi(t)dt$$

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$=\int_0^{min\{M(P_x,Q_z,t),[1.1],M(P_x,Q_z,t),M(P_x,Q_z,t)\}} \phi(t)dt = \int_0^{M\,(P_x,Q_z,t)} \phi(t)dt$

Therefore $w = z, z$ is a common fixed point of P, Q, S and T

Theorem (3.3): Let $(X, M,*)$ be a complete fuzzy metric space and let P, Q, S and T are self-mappings of X. Let the pairs {P, S} and {Q, T} be owc. If there exists q ∈(0, 1) such that

$\int_0^{M\,(P_x,Q_y,qt)} \phi(t)dt \geq \int_0^{\psi(M(S_x,P_x,t),M(T_y,P_x,t),M(P_x,T_y,2t),M(Q_y,P_x,3t)\}} \phi(t)dt$…. (2)

For all $x, y \in X$ and $t > 0$, $\phi: R^+ \to R$ is Lebesgue-integrable and increasing mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for all $\varepsilon > 0$,

$\psi: [0,1]^4 \to [0,1], \psi(1,t,t,t) > t$, and $t * t \geq t$.

Then P, Q, S and T have a unique common fixed point.

Proof: Since pairs of mappings {P, S} and {Q, T} are owc, there exists two points $x, y \in X$ such that $P_x = Q_y$

$\int_0^{M\,(P_x,Q_y,qt)} \phi(t)dt \geq \int_0^{\psi(M(S_x,P_x,t),M(T_y,P_x,t),M(P_x,T_y,2t),M(Q_y,P_x,3t)\}} \phi(t)dt$ by (2)

$$\geq \int_0^{\psi(1,M(P_x,Q_y,t),M(P_x,Q_y,t)*M(Q_y,T_y,t),M(Q_y,P_x,t)*M(P_x,P_x,t)*M(P_x,T_y,t)\}} \phi(t)dt$$

$= \int_0^{\psi(1,M(P_x,Q_y,t),M(P_x,Q_y,t)*1,M(P_x,Q_y,t)*1*M(P_x,Q_y,t)\}} \phi(t)dt$

$\geq \int_0^{\psi(1,M(P_x,Q_y,t),M(P_x,Q_y,t),M(P_x,Q_y,t)\}} \phi(t)dt > \int_0^{M\,(P_x,Q_y,t)} \phi(t)dt$

$\Rightarrow P_x = Q_y$ then $P_x = S_x = Q_y = T_y$

Suppose that z such that $P_z = S_z$ then by (2) we have $P_z = Q_z \Rightarrow P_z = S_z = Q_y = T_y$ so $P_x = P_z$ and $w = P_x = S_x$ is the unique point of P and S. Similarly there is $z \in X$ such that $z = Q_z = T_z$.

Assume $z \neq w$ ⇒by (2) we have

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$\int_0^{M\,(w,z,qt)} \phi(t)dt = \int_0^{M\,(P_x,Q_z,qt)} \phi(t)dt \geq$
$\int_0^{\psi(M(S_x,P_x,t),M(T_z,P_x,t),M(P_x,T_z,2t),M(Q_z,P_x,3t))} \phi(t)dt \geq \int_0^{M\,(P_x,Q_z,t)} \phi(t)dt$

Therefore $w = z$, z is a common fixed point of P, Q, S and T.

Theorem (3.4): Let $(X, M,*)$ be a complete fuzzy metric space and let P, Q, S and T are self-mappings of X. Let the pairs {P, S} and {Q, T} be owc. If there exists q ∈(0, 1) such that $\int_0^{M\,(P_x,Q_y,qt)} \phi(t)dt \geq$
$\int_0^{min\{M(S_x,T_y,t),M(P_x,S_x,t).M(Q_y,T_y,t),\psi[M(P_x,T_y,t),M(Q_y,S_x,2t)]\}} \phi(t)dt$…. (3)

For all $x, y \in X$ and $t > 0$, $\phi: R^+ \to R$ is Lebesgue-integrable and increasing mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for all

$\varepsilon > 0, \psi: [0, 1]^2 \to [0, 1], \psi(t, t) = t$ .Then P, Q, S and T have a unique common fixed point.

Proof: Since pairs of mappings {P, S} and {Q, T} are owc, there exists two points $x, y \in X$ such that $P_x = Q_y$

$\int_0^{M\,(P_x,Q_y,qt)} \phi(t)dt \geq$

$\int_0^{min\{M(S_x,T_y,t),M(P_x,S_x,t).M(Q_y,T_y,t),\psi[M(P_x,T_y,t),M(Q_y,S_x,2t)]\}} \phi(t)dt$ by (3)

$\geq \int_0^{min\{M(P_x,Q_y,t),1.1,\psi[M(P_x,T_y,t),M(Q_y,S_x,t)*M(S_x,S_x,t)]\}} \phi(t)dt$

$= \int_0^{min\{M(P_x,Q_y,t),\psi[M(P_x,Q_y,t),M(P_x,Q_y,t)*1]\}} \phi(t)dt$

$= \int_0^{min\{M(P_x,Q_y,t),M(P_x,Q_y,t)\}} \phi(t)dt = \int_0^{M\,(P_x,Q_y,t)} \phi(t)dt$

$\Rightarrow P_x = Q_y$ then $P_x = S_x = Q_y = T_y$

Suppose that z such that $P_z = S_z$ then by (3) we have $P_z = Q_z \Rightarrow P_z = S_z = Q_y = T_y$ so $P_x = P_z$ and $w = P_x = S_x$ is the unique point of P and S. Similarly there is $z \in X$ such that $z = Q_z = T_z$.

Assume $z \neq w \Rightarrow$by (3) we have

$\int_0^{M(w,z,qt)} \phi(t)dt = \int_0^{M(P_x,Q_z,qt)} \phi(t)dt \geq$

$\int_0^{min\{M(S_x,T_z,t),M(P_x,S_x,t).M(Q_z,T_z,t),\psi[M(P_x,T_z,t),M(Q_z,S_x,2t)]\}} \phi(t)dt \geq \int_0^{M(P_x,Q_z,t)} \phi(t)dt$

Therefore $w = z$, z is a common fixed point of P, Q, S and T.

Theorem (3.5): Let $(X, M,*)$ be a complete fuzzy metric space and let P, Q, S and T are self-mappings of X. Let the pairs {P, S} and {Q, T} be owc. If there exists q ∈(0, 1) such that $\int_0^{M(P_x,Q_y,qt)} \phi(t)dt \geq$

$$\int_0^{\alpha[M(P_x,Q_y,t)+M(P_x,T_y,t)]+\beta\psi[M(P_x,T_y,t),M(Q_y,S_x,t),M(S_x,P_x,t)]+\gamma[\min\{M(T_y,Q_y,t),M(Q_y,P_x,t),M(T_y,P_x,t)\}+\psi[M(S_x,Q_y,t)M(Q_y,S_x,t),M(P_x,S_x,t)]} \phi(t)dt \ldots(4)$$

For all $x, y \in X$ and $t > 0$, $\phi: R^+ \to R$ is Lebesgue-integrable and increasing mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for all $\varepsilon > 0, \psi: [0,1]^3 \to [0,1], \psi(t,t,1) = t$ and $\alpha, \beta, \gamma > 0$, $2\alpha + \beta + 2\gamma > 1$.Then P, Q, S and T have a unique common fixed point.

Proof: Since pairs of mappings {P, S} and {Q, T} are owc, there exists two points $x, y \in X$ such that $P_x = Q_y$

$$\int_0^{M(P_x,Q_y,qt)} \phi(t)dt \geq$$

$$\int_0^{\alpha[M(P_x,Q_y,t)+M(P_x,Q_y,t)]+\beta\psi[M(P_x,Q_y,t),M(P_x,Q_y,t),1]+\gamma[\min\{1,M(Q_y,P_x,t),M(P_x,Q_y,t)\}+\psi[M(P_x,Q_y,t)M(P_x,Q_y,t),1]} \phi(t)dt \quad \text{by (4)}$$

$$= \int_0^{\alpha[2M(P_x,Q_y,t)]+\beta M(P_x,Q_y,t)+\gamma[M(P_x,Q_y,t)+M(P_x,Q_y,t)]} \phi(t)dt$$

$$= \int_0^{2\alpha[M(P_x,Q_y,t)]+\beta M(P_x,Q_y,t)+2\gamma[M(P_x,Q_y,t)]\}} \phi(t)dt = \int_0^{(2\alpha+\beta+2\gamma)M(P_x,Q_y,t)} \phi(t)dt$$

$$\geq \int_0^{M(P_x,Q_y,t)} \phi(t)dt \quad \Rightarrow P_x = Q_y \text{ then } P_x = S_x = Q_y = T_y$$

Suppose that z such that $P_z = S_z$ then by (3) we have $P_z = Q_z \Rightarrow P_z = S_z = Q_y = T_y$ so $P_x = P_z$ and $w = P_x = S_x$ is the unique point of P and S. Similarly there is $z \in X$ such that $z = Q_z = T_z$.

Assume $z \neq w$ ⇒by (3) we have

$\int_0^{M\,(w,z,qt)} \phi(t)dt = \int_0^{M\,(P_x,Q_z,qt)} \phi(t)dt \geq$

$\int_0^{\substack{\alpha[M\,(P_x,Q_z,t)+M\,(P_x,Q_z,t)]+\beta\psi[M\,(P_x,Q_z,t),M\,(P_x,Q_z,t),1]+ \\ \gamma[\min\{1,M\,(Q_z,P_x,t),M\,(P_x,Q_z,t)\}+\psi[M\,(P_x,Q_z,t)M\,(P_x,Q_z,t),1]}} \phi(t)dt$ by (4)

$\geq \int_0^{M\,(P_x,Q_z,t)} \phi(t)dt \quad \Rightarrow P_x = Q_z$ then $P_x = S_x = Q_z = T_z$

Therefore $w = z$, z is a common fixed point of P, Q, S and T.

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