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Generalized Fourier-Laplace Transform and Differential Equations

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Abstract— Integral transformations are very important tool for solving various problems in mathematics, physics and engineering. Fourier and Laplace transforms are powerful tools for image processing and signal processing. However we have combined these transforms to form Fourier-Laplace transform and tried to solve different types of problems. This paper gives the application of generalized Fourier-Laplace transform for solving some differential equations. Also we introduced new differential operator and its adjoint operator and solved the Differential equations using Fourier-Laplace transform.

Keywords- *Fourier transform, Laplace transform, Fourier-Laplace transform, Generalized function, Adjoint Operator.*

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1. INTRODUCTION

The importance of Integral Transforms is that they provide powerful operational methods for solving many problems from many different fields [1]. Presently Integral transforms are knocking the door of many new subjects. We find Integral Transform as a problem solver not only in the areas of differential and difference equations, electric circuits and networks, vibration and wave propagation, heat conduction in solids, quantum mechanics, fractional calculus and fractional differential equations, dynamic systems, but also in signal processing, integral equations, physical chemistries mathematical biology, probability and statistics and solid and fluid mechanics and in Medical sciences also [2], [3].

Here the Fourier Transform is used as a tool for solving many linear boundary value and initial value problems in Applied Mathematics, Mathematical physics and Engineering science. It also have a wide scope covering these areas- Distributions and generalized transforms, Convolutions, and correlations and applications, probability distributions, sampling theory, filters, and analysis of linear systems, nuclear magnetic resonance (NMR) and in other kinds of spectroscopy and the Dirac delta [1].

The Laplace transform is a widely used integral transform with many applications in Physics and Engineering. The Laplace Transform is a simple way of converting functions from one domain to the functions of another domain. Moreover, the Laplace transform can also be used to solve differential equations and it is used in Electrical Engineering, too. For Example, the Laplace Transform is widely used in electrical circuits for the analysis of linear time-invariant systems [4]. The Laplace transform is really just a shortcut for complex calculations. It may seem troublesome, but it bypasses some of the most difficult mathematics.

Many authors studied on various integral transforms separately. However there is much scope in extending double transformation to a certain class of generalized functions. Bhosale B. N. and Choudhary M.S. [5] and Khairnar S. M. et.al. [6] has discussed double transform and their application. Motivated by this we have also defined a new combination of integral transforms in distributional generalized sense namely Fourier-Laplace transform and its applications to some particular types of differential equations by introducing a new differential operator and its Adjoint operator. For this it is necessary to define the conventional Fourier-Laplace transform which is defined as follows:

$$FL\{f(t,x)\} = F(s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f(t,x)K(t,x)\,dt\,dx \ ,$$

where, $K(t,x) = e^{-i(st-ipx)}$.

Also we have defined the testing function spaces as well as distributional generalized Fourier-Laplace transform in our previous work, which are given as follows:

1.1. The space $FL_{a,b,\alpha}$

This space is given by

$$FL_{a,b,\alpha} = \left\{ \phi : \phi \in E_{+} \,/\, \xi_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^{k} e^{ax} D_t^{l} D_x^{q} \phi(t,x) \right| \le C_{lq} A^{k} k^{k\alpha} \right\} \quad (1.1.1)$$

Where the constants A and C_{lq} depend on the testing function ϕ .

1.2. Distributional Generalized Fourier-Laplace Transforms (FLT)

For $f(t,x) \in FL^{*\beta}_{a,\alpha}$, where $FL^{*\beta}_{a,\alpha}$ is the dual space of $FL^{\beta}_{a,\alpha}$. It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\{f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle \quad (1.2.1)$$

,

where, for each fixed t $(0<t<\infty)$, x $(0<x<\infty)$, $s>0$ and $p>0$, the right hand side of (1.2.1) has a sense as an application of $f(t,x) \in FL^{*\beta}_{a,\alpha}$ to $e^{-i(st-ipx)} \in FL^{\beta}_{a,\alpha}$.

The paper is organized as follows:

In section 2 new differential operator is introduced. In section 3 another operator is defined which is the adjoint operator of the operator defined in section 2. Section 4 gives applications of generalized Fourier-Laplace Transform to differential equations. Section 5 concludes the paper.
Notations and terminology used as in Zemanian [7], [8].

2. Application of Fourier-Laplace Transform

The kernel of Fourier-Laplace transform is

$$K(t,x,s,p)=e^{-i(st-ipx)},$$

Differentiate with respect to t and x, we get

$$D_tD_xK(t,x,s,p)=(-is)(-p)e^{-i(st-ipx)}$$
$$=ispe^{-i(st-ipx)}$$
$$=ispK(t,x,s,p)$$

We construct an operator $\Lambda_{t,x}=D_tD_x+isp$,

where $D_t=\frac{d}{dt}$, $D_x=\frac{d}{dx}$

$$\Lambda_{t,x}K(t,x,s,p)=[D_tD_x+isp]K(t,x,s,p)$$

$$=D_tD_xK(t,x,s,p)+ispK(t,x,s,p)$$

$$=ispK(t,s,s,p)+ispK(t,x,s,p)$$
$$=2ispK(t,s,s,p)$$
$$=C_0K(t,s,s,p),$$

where $C_0=2isp$ is constant

Continuing in the same way, we have

$$\Lambda_{t,x}^2K(t,x,s,p)=(C_0)^2K(t,x,s,p)$$

$$\Lambda_{t,x}^3K(t,x,s,p)=(C_0)^3K(t,x,s,p)$$

$$\Lambda_{t,x}^4K(t,x,s,p)=(C_0)^4K(t,x,s,p)$$

And so on,

$$\Lambda_{t,x}^kK(t,x,s,p)=(C_0)^kK(t,x,s,p)=(2isp)^kK(t,x,s,p)$$

Since the operator

$\Lambda_{t,x}^kK(t,x,s,p)=(2isp)^kK(t,x,s,p)$ is obviously linear and continuous. We have

$$FL\{\Lambda_{t,x}^k[f(t,x)]\}=\langle\Lambda_{t,x}^k[f(t,x)],K(t,x,s,p)\rangle$$
$$=\langle f(t,x),\Lambda_{t,x}^kK(t,x,s,p)\rangle$$

$$=\langle f(t,x),(2isp)^kK(t,x,s,p)\rangle$$

Therefore we have,

$$FL\{\Lambda_{t,x}^k[f(t,x)]\}=\langle f(t,x),((C_0)^k)K(t,x,s,p)\rangle$$

for all $f\in FL_{a,\alpha}^{*\beta}$.

3. Adjoint Operator $\Lambda_{t,x}^*$

We define an operator $\Lambda_{t,x}^*:FL_{a,\alpha}^{*\beta}\to FL_{a,\alpha}^{\beta}$ using the relation

$$\langle\Lambda_{t,x}^*(f(t,x)),\phi(t,x)\rangle=\langle f(t,x),\Lambda_{t,x}(\phi(t,x))\rangle$$

For all $f\in FL_{a,\alpha}^{*\beta}$ and $\phi\in FL_{a,\alpha}^{\beta}$. The operator $\Lambda_{t,x}^*$ is called the adjoint operator of $\Lambda_{t,x}$ .For each $k=1,2,3,........$ we easily get

$$\langle(\Lambda_{t,x}^*)^k(f(t,x)),\phi(t,x)\rangle=\langle f(t,x),(\Lambda_{t,x}^*)^k(\phi(t,x))\rangle$$

It can be readily shown that if f regular distribution generated by an element in then $\Lambda_{t,x}^*f=\Lambda_{t,x}f$ for each $k=1,2,3,......$ we have

$$\langle(\Lambda_{t,x}^*)^k(f(t,x)),K(t,x,s,p)\rangle=\langle f(t,x),(\Lambda_{t,x}^*)^kK(t,x,s,p)\rangle$$
$$=\langle f(t,x),(2isp)^kK(t,x,s,p)\rangle$$

$$=\langle f(t,x),((C_0)^k)K(t,x,s,p)\rangle$$

$$=((C_0)^k)\langle f(t,x),K(t,x,s,p)\rangle$$

Thus we arrive at the important result, for each $k=1,2,3,......$ we have for $f\in FL_{a,\alpha}^{*\beta}$

$$FL\{(\Lambda_{t,x}^*)^k(f(t,x))\}=(C_0)^kFL\{f(t,x)\}(s,p)$$

4. AN APPLICATION OF THE FOURIER-LAPLACE TRANSFORM TO DIFFERENTIAL EQUATIONS

4.1. Solution of $P(\Lambda_{t,x}^*)u=f$

Consider the differential equation $P\left(\Lambda^*_{t,x}\right)u = f$ (4.1.1)

Where $f \in FL^{*\beta}_{a,\alpha}$ and P any polynomial of degree m. Suppose that the equation (4.1.1) possesses a solution u.

Applying the Fourier-Laplace transform to (4.1.1) we have,

$FL\left[P\left(\Lambda^*_{t,x}\right)u\right] = FL(f)$ if $FL[f] = f^\wedge$ then

$$P(2isp)FL\left[u(t,x)\right] = f^\wedge$$

using $FL\left[\left(\Lambda^*_{t,x}\right)^k\left(f(t,x)\right)\right] = (2isp)^k FL\{f(t,x)\}$

$\Rightarrow P(2isp)u^\wedge = f^\wedge$ i.e.

$$P\left(\left(C_0\right)\right)u^\wedge = f^\wedge \tag{4.1.2}$$

where $u^\wedge = FL\left[u(t,x)\right]$

If we further assume that the polynomial P is such that for $\in > 0$

$$\left|P(2isp)\right| < \in \tag{4.1.3}$$

Then under this assumption (4.1.2) gives

$$u^\wedge = \left[P(2isp)\right]^{-1} f^\wedge \tag{4.1.4}$$

Applying inversion of Fourier-Laplace transform, we get

$$u = (FL)^{-1}\left[\frac{f^\wedge}{P(2isp)}\right] = (FL)^{-1}\left[\frac{f^\wedge}{P(C_0)}\right]$$

Hence the proof.

4.2. Solution of differential equation $P(D)u = f$

Consider the differential equation $P(D)u = f$ (4.2.1)

When $f \in FL^{*\beta}_{a,\alpha}$ and $P(D) = \sum_{|\beta| \le m} a_\beta D^\beta$

Is a linear differential operator of order m with constant coefficients.

Suppose that the equation (4.2.1) possesses a solution u.

Applying the Fourier-Laplace transform to (4.2.1) and using

$$D^l_t D^q_x K(t,x,s,p) = (-is)^l (-p)^q e^{-i(st-ipx)}$$

$$= (-is)^l (-p)^q K(t,x,s,p)$$

$$= (-1)^{l+q} (is)^l (p)^q K(t,x,s,p)$$

We have $FL\left[P(D)u\right] = FL[f] = f^\wedge$ (say) (4.2.2)

We can reform them to the Fourier-Laplace transform and hence we get

$$P(t,x,s,p)u^\wedge = f^\wedge \tag{4.2.3}$$

where $P(t,x,s,p)$ is polynomial in $t,x,s,$ and p, $u^\wedge = FL[u]$

Under the assumption that the polynomial P is such that

$$P\left(D^l_t D^q_x K(t,x,s,p)\right) > \xi > 0$$

for $\xi_1, \xi_2, \xi_3, \ldots\ldots\ldots\xi_n \in R^n$.

$$u^\wedge = P\left[D^l_t D^q_x K(t,x,s,p)\right]^{-1} f^\wedge \tag{4.2.4}$$

Applying inversion of Fourier-Laplace transform to above equation, we have

$$u = (FL)^{-1}\left[\frac{f^\wedge}{P\left(D^l_t D^q_x K(t,x,s,p)\right)}\right]$$

5. CONCLUSION

In the present work we have introduced new operator Λ and its adjoint operator Λ^*. And using that the differential equations is solved for Fourier-Laplace transform.

REFERENCES

[1] Lokenath Debnath, and Dambaru Bhatta, "Integral Transforms and their Applications," Chapman and Hall/CRC Taylor and Francis Group Boca Raton London, New York, 2007.

[2] Joon Eom Hyo, 'Integral transforms in Electromagnetic Formulation,' Journal of Electromagnetic Engineering and science, Vol. 14, No. 3, pp. 273-277, Sept. 2014.

[3] Poonia Sarita and Mathur Rachna Comparative Study of Double Mellin-Laplace and Fourier Transform, International Journal of Mathematics Research, Vol.5, No.1, pp. 41-48, (2013).

[4] M. C. Anumaka, "Analysis and applications of Laplace/Fourier transformations in electric circuit," IJRRAS, 12(2) pp. 333-339, 2012.

[5] B. N. Bhosale, and M. S. Chaudhary, "Fourier-Hankel Transform of Distribution of compact support," J. Indian Acad. Math, 24(1), pp. 169-190, 2002.

[6] S.M. Khairnar, R.M. Pise, and J. N. Salunke, "Applications of the Laplace-Mellin integral transform to differential equations," International Journal of Scientific and Research Publications, 2(5) pp. 1-8, (2012).

[7] A. H. Zemanian, "Generalized integral transform," Inter science publisher, New York, 1968.

[8] A. H. Zemanian, "Distribution theory and transform analysis," McGraw Hill, New York, 1965.

[9] B. N. Bhosale, "Integral transformations of generalized functions," Discovery Publishing House, New Delhi, (2005).

[10] W.T. Thomson, "Laplace Transformation" (2nd Edition), Prentice-Hall, 1960.

[11] V. D. Sharma, "Operation Transform Formulae on Generalized Fractional Fourier Transform," Proceedings International Journal of Computer Applications (IJCA), (0975-8887), PP. 19-22, 2012.

[12] V. D. Sharma, and A.N. Rangari, "Operation Transform Formulae of Fourier-Laplace Transform," Int. Journal of Pure and Applied Sciences and Technology, 15(2), pp. 62-67, (2013).

[13] V.D. Sharma, and A.N. Rangari, "Operational Calculus on Generalized Fourier-Laplace Transform," Int. Journal of Scientific and Innovative Mathematical Research (IJSIMR), 2(11), pp. 862-867, (2014).

[14] V. D. Sharma, and S.A. Khapre, "Applications on Generalized 2D Fractional cosine transform," Int. J. of Engineering and Innovative technology, Vol. 3. No. 4, pp. 139-143, Oct. 2013.

[15] V.D. Sharma, and S.A. Khapre, "Some examples on generalized two-dimensional fractional cosine transform in the range $-\infty$ to $+\infty$," International Journal of science and research, Vol. 3, Issue 8, pp. 1199-1202, August 2014.

[16] R. J. Beerends, H. G. ter Morsche, J. C. van den Berg, and E. M. van de Vrie, "Fourier and Laplace Transforms," Cambridge University Press, 2003.

[17] Patrick Fitzsimmons, and Tucker Mc EL Roy, "On Joint Fourier-Laplace Transforms," Communication in statistics-Theory and Methods, 39: 1883-1885 Taylor and Francis Group, LLC, 2010.

[18] V. D. Sharma, and A. N. Rangari, "Representation theorem for the Distributional Fourier-Laplace Transform," Int. Journal of Science and Research (IJSR), Vol. 3, Issue 8, pp. 341-344, August 2014.