Application of Two Dimensional Fractional Sine Transforms to Differential Equation

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***Abstract*—**Transforms with cosine and sine functions as the transform kernels represent an important area of analysis. It is based on the so-called half-range expansion of a function over a set of cosine or sine basis functions. Because the cosine and the sine kernels lack the nice properties of an exponential kernel, many of the transform properties are less elegant and more involved than the corresponding ones for the Fourier transform kernel. The aim of this paper is we introduced ne w differential operator and also its ad joint operator

Keywords- *fractional Fourier transform, fractional Cosine transform, fractional Sine transform.*

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I. INTRODUCTION:

Various transforms are employed for signal processing to obtain useful information, which is not explicitly available when the signal is in the time domain. Most of the real time signals such as speech, biomedical signals, etc., are non-stationary signals.

The Fourier transform (FT), used for most of the signal processing applications, determines the frequency components present in the signal but with zero time resolution. The fractional transforms, such as Fractional Fourier Transform(FRFT) [1–3], Fractional Cosine Transform (FRCT) [4–6] or fractional Sine Transform (FRST) [4,5] describe the energy density or signal intensity simultaneously in the time and frequency domain and have non-zero time frequency resolution in the transform domain. These transforms are used for optical signal processing, time variant filtering, as swept frequency filters, for pattern recognition and signal compression [1–3]. In real time applications, samples of input signal arrive in a sequential manner. The transform of the signal is obtained by processing sequentially the blocks of N samples. In this procedure, for the computation of transform of N input samples, the system has to wait till the arrival of all N new input samples. Instead, when a new sample arrives the transform can be computed by processing the new block of samples consisting of the newly arrived sample and the N-1 samples of the previous block. This technique is referred to as the sliding technique. Since the fractional transforms have non-zero time frequency resolution, they are better suited for processing the non-stationary real time signals. FRFT is a generalization of ordinary FT with an order parameter ¼ a p/2. The order parameter a represents the angle of rotation of the signal in time frequency plane. The FRFT is identical to ordinary FT when a ¼ p/2 [1–3]. The fractional Fourier transform with its kernel expressed in the closed form has the lowest complexity and negligible errors in computation satisfy many of the properties. It suited for most of the real time applications due to the simpler and closed form of fractional convolution and correlation [9]. The fractional cosine transforms and the fractional sine transform are related to [4, 5].

1.1. Generalized two dimensional fractional Sine transform

Two dimensional fractional sine transform with parameter α f(x, y) denoted by F_α^S perform a linear operation given by the integral transform.

$$F_s^\alpha\{f(x,y)\}(u,v) = \int_0^\infty \int_0^\infty f(x,y) K_\alpha(x,y,u,v)\,dx\,dy \dots\dots\dots \quad (1.1)$$

Where the kernel,

$$K_s^\alpha(x,y,u,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i(x^2+y^2+u^2+v^2)\cot\alpha}{2}} e^{i(\frac{\pi}{2}-\alpha)} \sin(\operatorname{cosec}\alpha\,.\,ux)\,.\sin(\operatorname{cosec}\alpha\,.\,vy)$$

………..……… (1.2)

1.2. The test function space E

An infinitely differentiable complex valued function φ on R^n belongs to $E(R^n)$ if for each compact set I $\in S_{a,b}$, where,

$S_{a,b} = \{x,y : x,y \in R^n, |x| \le a, |y| \le b, a > 0, b > 0\}, I \in R^n$

$$\gamma_{E_{p,q}}(\emptyset) = \sup_{x,y} \left|D_{x,y}^{p,q}\emptyset(x,y)\right| < \infty$$

Where, p, q =1, 2, 3….

Thus $E(R^n)$ will denote the space of all $\varphi \in E(R^n)$ with support contained in $S_{a,b}$

Note: that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Cosine transformable, if it is a member of , the dual space of E.

In the present work, Generalization of two dimensional fractional sine transform is presented

The new adjoint operator is defined. And using it the differential equation is solved.

II. Distributional two-dimensional fractional Sine transform

The two dimensional distributional fractional Sine transform of defined by

$$F_s^\alpha\{f(x,y)\} = F^\alpha(u,v)$$
$$= \langle f(x.$$

2.1)

$$K_s^\alpha(x,y,u,v) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i(x^2+y^2+u^2+v^2)cot\alpha}{2}} e^{i\left(\frac{\pi}{2}-\alpha\right)}$$
$$\sin(cosec\alpha . ux) . \sin \ldots.. \qquad (2.2)$$

Where , RHS of equation (2.1) has a meaning as the application of to

III. APPLICATION OF FRACTIONAL SINE TRANSFORM

Kernel of two dimensional fractional cosine transform as

$$K_s^\alpha(x,y,u,v) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i(x^2+y^2+u^2+v^2)cot\alpha}{2}}$$
$$e^{i\left(\frac{\pi}{2}-\alpha\right)}\sin(cosec\alpha . ux) . \sin(co$$

We can arrange as,

$$K_s^\alpha(x,y,u,v) = A\, e^{\frac{i(x^2+y^2)cot\alpha}{2}}\sin(Px) . \sin(Qy)$$

Where $A = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i(x}{}}$

$P = cosec\alpha . u \quad Q = cosec\alpha . v$

$$D_xD_yK_s^\alpha(x,y,u,v) = D_xD_yA\, e^{\frac{i(x^2+y^2)cot\alpha}{2}}\sin(Px) . \sin(Qy)$$

$$D_xD_yK_s^\alpha(x,y,u,v)$$
$$= AD_x\left(e^{\frac{i(x^2)cot\alpha}{2}}\sin(Px)\right)D_y\left(e^{\frac{i(y^2)cot\alpha}{2}}\sin(Qy)\right)$$

$$D_xD_yK_s^\alpha(x,y,u,v)$$
$$= A\left[PcosPx\, e^{\frac{i(x^2)cot\alpha}{2}} + ixcot\alpha . sinPx\, e^{\frac{i(x}{}}\right.$$

$$\left[QcosQy e^{\frac{i(y^2)cot\alpha}{2}} + iycot\alpha . sinQy e^{\frac{i(y^2)cot\alpha}{2}}\right]$$

$$D_xD_yK_s^\alpha(x,y,u,v) = Ae^{\frac{i(x^2+y^2)cot\alpha}{2}}$$
$$[PcosPx + ixcot\alpha . sinPx][QcosQy + iycot\alpha . sinQy]$$

$$D_xD_yK_s^\alpha(x,y,u,v) = Ae^{\frac{i(x^2+y^2)cot\alpha}{2}} sinPxsinQy$$
$$[PcotPx + ixcot\alpha][QcotQy + iycot\alpha]$$

$$D_xD_yK_s^\alpha(x,y,u,v) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i(x^2+y^2+u^2+v^2)cot\alpha}{2}} e^{i\left(\frac{\pi}{2}-\alpha\right)}$$
$$\sin(cosec\alpha . ux)\ \text{si}$$
$$\left\{\begin{matrix}[cosec\alpha . u\ cot(cosec\alpha . ux) + ixcot\alpha] \\ [cosec\alpha . vcot(cosec\alpha . vy) + iycot\alpha]\end{matrix}\right\}$$

$$D_xD_yK_s^\alpha(x,y,u,v) = K_s^\alpha(x,y,u,v)$$
$$\left\{\begin{matrix}[cosec\alpha . u\ cot(cosec\alpha . ux) + ixcot\alpha] \\ [cosec\alpha . vcot(cosec\alpha . vy) + iycot\alpha]\end{matrix}\right\}$$

$$D_xD_yK_s^\alpha(x,y,u,v) = K_s^\alpha(x,y,u,v)$$

$$\left\{\begin{matrix} i^2xycot^2\alpha + ixcot\alpha csc\alpha . vcot(cosec\alpha . vy) \\ +icosec\alpha . ucot(cosec\alpha . ux)ycot\alpha \\ +csc^2\alpha . u . v . cot(cosec\alpha . ux)cot(cosec\alpha . vy)\end{matrix}\right\}$$

$$D_xD_yK_s^\alpha(x,y,u,v) = K_s^\alpha(x,y,u,v)$$
$$\left\{\begin{matrix} -xycot^2\alpha + ixcot\alpha csc\alpha . vcot(cosec\alpha . vy) \\ +icosec\alpha . ucot(cosec\alpha . ux)ycot\alpha \\ +csc^2\alpha . u . v . cot(cosec\alpha . ux)cot(cosec\alpha . vy)\end{matrix}\right\}$$

$$\bigwedge_{x,y} = x^{-1}y^{-1}D_xD_y -$$
$$\left\{\begin{matrix}\left(cot^2\alpha + \frac{icot\alpha csc\alpha . v . \cot(csc\alpha . vy)}{y}\right) \\ + \frac{icsc\alpha . u . ycot\alpha\cot(csc\alpha . ux)}{x} \\ + \frac{csc^2\alpha u . v\cot(csc\alpha . ux)\cot(csc\alpha . vy)}{xy}\end{matrix}\right\}$$

$$\bigwedge_{x,y} K_s^\alpha(x,y,u,v) = x^{-1}y^{-1}D_xD_yK_c^\alpha(x,y,u,v) -$$
$$\left\{\begin{matrix}\left(cot^2\alpha + \frac{icot\alpha csc\alpha . v . \cot(csc\alpha . vy)}{y}\right) \\ + \frac{icsc\alpha . u . ycot\alpha\cot(csc\alpha . ux)}{x} \\ + \frac{csc^2\alpha u . v\cot(csc\alpha . ux)\cot(csc\alpha . vy)}{xy}\end{matrix}\right\} K_s^\alpha(x,y,u,v)$$

$$\bigwedge_{x,y} K_s^\alpha(x,y,u,v) = x^{-1}y^{-1}$$
$$\left(\begin{matrix} K_s^\alpha(x,y,u,v) \\ \left\{\begin{matrix} -xycot^2\alpha \\ +ixcot\alpha csc\alpha . vcot(cosec\alpha . vy) \\ +icosec\alpha . ucot(cosec\alpha . ux)ycot\alpha \\ +csc^2\alpha . u . v . cot(cosec\alpha . ux)cot(cosec\alpha . vy)\end{matrix}\right\}\end{matrix}\right) -$$
$$\left\{\begin{matrix}\left(cot^2\alpha + \frac{icot\alpha csc\alpha . v . \cot(csc\alpha . vy)}{y}\right) \\ + \frac{icsc\alpha . u . ycot\alpha\cot(csc\alpha . ux)}{x} \\ + \frac{csc^2\alpha u . v\cot(csc\alpha . ux)\cot(csc\alpha . vy)}{xy}\end{matrix}\right\} K_s^\alpha(x,y,u,v)$$

$$\bigwedge_{x,y} K_s^\alpha(x,y,u,v)$$
$$= \left\{\begin{matrix} x^{-1}y^{-1}\left\{\begin{matrix} -xycot^2\alpha + ixcot\alpha csc\alpha . vcot(cosec\alpha . vy) \\ +icosec\alpha . ucot(cosec\alpha . ux)ycot\alpha \\ +csc^2\alpha . u . v . cot(cosec\alpha . ux)cot(cosec\alpha . vy)\end{matrix}\right\} \\ -\left\{\begin{matrix}\left(cot^2\alpha + \frac{icot\alpha csc\alpha . v . \cot(csc\alpha . vy)}{y}\right) \\ + \frac{icsc\alpha . u . ycot\alpha\cot(csc\alpha . ux)}{x} \\ + \frac{csc^2\alpha u . v\cot(csc\alpha . ux)\cot(csc\alpha . vy)}{xy}\end{matrix}\right\}\end{matrix}\right\}$$
$$K_s^\alpha(x,y,u,v)$$

$$\bigwedge_{x,y} K_s^\alpha(x,y,u,v)$$

$$= \left\{ \begin{array}{c} -cot^2\alpha + \frac{icot\alpha csc\alpha . vcot(cosec\alpha . vy)}{y} \\ + \frac{icosec\alpha . u\, cot(cosec\alpha . ux)\, ycot\alpha}{x} \\ + \frac{csc^2\alpha . u . v . cot(cosec\alpha . ux) cot(cosec\alpha . vy)}{xy} \\ -cot^2\alpha \\ - \frac{icot\alpha csc\alpha . vcot(cosec\alpha . vy)}{y} \\ - \frac{icsc\alpha . u . ycot\alpha cot(csc\alpha . ux)}{x} \\ - \frac{csc^2\alpha u . vcot(csc\alpha . ux) cot(csc\alpha . vy)}{xy} \end{array} \right\}$$

$K_s^\alpha(x, y, u, v)$

$\wedge_{x,y} K_s^\alpha(x, y, u, v) = \{-2cot^2\alpha\} K_s^\alpha(x, y, u, v)$
$\wedge_{x,y} K_s^\alpha(x, y, u, v) = \{2i^2 cot^2\alpha\} K_s^\alpha(x, y, u, v)$
$\wedge_{x,y}{}^2 K_s^\alpha(x, y, u, v) = (2i^2 cot^2\alpha)^2 K_s^\alpha(x, y, u, v)$
$\wedge_{x,y}{}^2 K_s^\alpha(x, y, u, v) = (C_\alpha)^2 K_s^\alpha(x, y, u, v)$
Where
$\wedge_{x,y}{}^3 K_s^\alpha(x, y, u, v) = (C_\alpha)^3 K_s^\alpha(x, y, u, v)$
$\wedge_{x,y}{}^4 K_s^\alpha(x, y, u, v) = (C_\alpha)^4 K_s^\alpha(x, y, u, v)$
So on

$\wedge_{x,y}{}^k K_s^\alpha(x, y, u, v) = (C_\alpha)^k K_s^\alpha(x, y, u, v)$
Therefore we have
$F_\alpha^s\{\wedge_{x,y}^k f(x, y)\} = \langle f(x, y), (C_\alpha)^k K_s^\alpha(x, y, u, v)\rangle$
Foe all and for

IV. Ad joint Operator

We define an operator : using the relation
$\langle \wedge_{x,y}^* \{f(x, y)\}, \varphi(x, y)\rangle = \langle f(x, y), \wedge_{x,y} \varphi$ For all and
The operator is called the ad joint operator of for each k=1, 2, 3…
We easily get
$\langle (\wedge_{x,y}^*)^k \{f(x, y)\}, \varphi(x, y)\rangle = \langle f(x, y), (\wedge_{x,y}^*)^k \varphi(x, y)\rangle$
It can be readily shown that f is regular distribution generated by an element in then

For each k=1, 2, 3…… and for we have
$\langle (\wedge_{x,y}^*)^k \{f(x, y)\}, K_s^\alpha(x, y, u, v)\rangle$
$= \langle f(x, y), (\wedge_{x,y}^*)^k K_s^\alpha(x, y, u, v)\rangle$
$\langle (\wedge_{x,y}^*)^k \{f(x, y)\}, K_s^\alpha(x, y, u, v)\rangle = \langle f(x, y), (2i^2 cot^2\alpha)^k K_s^\alpha(x, y, u, v)\rangle$

$\langle (\wedge_{x,y}^*)^k \{f(x, y)\}, K_s^\alpha(x, y, u, v)\rangle$
$= \langle f(x, y), (C_\alpha)^k K_s^\alpha(x, y, u, v)\rangle$
$\langle (\wedge_{x,y}^*)^k \{f(x, y)\}, K_s^\alpha(x, y, u, v)\rangle$
$= (C_\alpha)^k \langle f(x, y), K_s^\alpha(x, y, u, v)\rangle$
Thus we arrive at the important result, for each k=1, 2, 3….. And for
We have for
$F_\alpha^s\left\{(\wedge_{x,y}^*)^k \{f(x, y)\}\right\} = (C_\alpha)^k F_\alpha^s\{\{f(x, y)\}\}(u, v)$

V. AN APPLICATION OF THE FRACTIONAL SINE TRANSFORM TO DIFFERENTIAL EQUATIONS OF

Solution: consider the differential equation …………… (1)
Where and P any polynomial of degree m.
Suppose that the equation (1) possesses a solution U.
Applying the fractional cosine transform to (1)
We have,
$F_\alpha^s[P(\wedge_{x,y}^*)U$ If then
$P(2i^2 cot^2\alpha) F_\alpha^s$ Using
$F_\alpha^s\left\{(\wedge_{x,y}^*)^k \{f(x, y)\}\right\} = (2i^2 cot^2\alpha)^k F_\alpha^s\{\{f(x, y)\}\}$
P
……………………………… (2)
Where
If we further assume that the polynomial P is such that for

For
Then under this assumption (2) gives
$U^\wedge = [P(2i^2 cot^2\alpha)]^{-1} f^\wedge$
Applying the inversion of fractional cosine transform we get
$U = (F_\alpha^s)^{-1}\left[\frac{f^\wedge}{P(2i^2 cot\alpha)}\right] = (F_\alpha^s$
Hence proof.

CONCLUSION:

In the present work, the new adjoint operator is defined. And using it the differential equation is solved.

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