Quantum and Floer Type Cohomologies on Almost Contact Metric Manifolds

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***Abstract*—** We study the relations between quantum type cohomologies and Floer type cohomologies on almost contact metric manifolds with a closed fundamental 2-form. For quantum type cohomologies on the manifolds, we investigated pseudo-coholomorphic curves, moduli spaces of the curves representing 2-dimensional homology classes, Gromov-Witten type invariants, and quantum type product on cohomology groups. For Floer type cohomologies on cosymplectic manifolds, we study a symplectic type action funtional on the universal covering space of the space of contractible loops. The critical points of the funtional and the moduli space of gradient flow lines joining critical points induce a cochain complex and produce a Floer type cohomology. We extend Floer type cohomology to more general almost contact manifolds and show that the quantum type cohomology and the Floer type cohomology are isomorphic.

Keywords- *almost contact metric manifold; Gromov-Witten type invariant; quantum type cohomology; Maslov index; symplectic type action fuctional; Floer type cohomology*

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I. Introduction

A symplectic manifold (M^{2n}, ω, J) is called semipositive if, for all $A \in \pi_2(M)$, $\omega(A) > 0$ and the dimension of moduli space $\mathcal{M}(A, J) \geqslant 0$, then $c_1(A) \geqslant 0$. Let (M^{2n}, ω, J) be a semipositive symplectic manifold and let $H_t = H_{t+1} : M \rightarrow \mathbb{R}$ be a smooth 1-periodic family of Hamiltonian functions. Let $X_t : M \rightarrow TM$ be the Hamiltonian vector field defined by $\omega(X_t, \bullet) = dH_t(\bullet)$ and consider the time dependent Hamiltonian differential equation $\dot{x}(t) = X_t(x(t))$.

Let $\widetilde{LM}$ be a unique covering space of the space LM of contractible loops in M. Where an element of $\widetilde{LM}$ is of the form $[x, u]$ in which $u : D^2 \rightarrow M$ is a smooth map from the unit disc D^2 to M such that $u(e^{2\pi it}) = x(t)$, and $[x, u_1] = [x, u_2]$ if and only if $u_1\#(-u_2)$ is homologous to zero in $H_2(M; \mathbb{Z})$. The $H_2(M)$ acts on $\widetilde{LM}$ as a deck transformation, via $(A, [x, u]) \mapsto A\#[x, u]$.

The symplectic action functional $a_H : \widetilde{LM} \rightarrow \mathbb{R}$ is defined by

$$a_H([x, u]) = -\int_{D^2} u^* \omega - \int_0^1 H_t(x(t))dt$$

and satisfies $a_H(A\#[x, u]) = a_H([x, u]) - \omega(A)$. The critical points of a_H are the equivalence classes $[x, u]$ which are contractible periodic solution of $\dot{x}(t) = X_t(x(t))$. The function can be interpreted as a closed 1-form on the LM rather than a function on the covering space $\widetilde{LM}$ which is the situation considered by Novikov.

The gradient flow lines of a_H are solutions

$$u : \mathbb{R} \rightarrow M : (s, t) \mapsto u(s, t)$$

of the partial differential equation

$$\partial_s u + J(u)(\partial_t u - X_t(u)) = 0$$

with periodic condition $u(s, t+1) = u(s, t)$ and limit condition $\lim_{s\rightarrow\pm\infty} u(s, t) = x^{\pm}(t)$ where $x^{\pm} \in LM$ are in the images of the projection of the critical points of a_H.

The critical points and the Floer connecting orbits determine a cochain complex (CF^*, δ), and its cohomology groups

$$HF^*(M, \omega, H, J) = \frac{\ker \delta}{\operatorname{im} \delta}$$

are called the Floer cohomology groups of M for a regular pair (H, J). There is a natural ring structure on Floer cohomology which is defined by counting perturbed J-holomorphic curves on the 2-sphere called the pair-of-pant product.

In section II we recall almost contact metric manifolds with either cosymplectic, contact, or C-manifold structure.

In section III we introduce the theory of pseudo-coholomorphic curves in almost contact metric manifolds with a closed fundamental 2-form, Gromov-Witten type invariants, and quantum type cohomologies.

In section IV we study the symplectic type action functional defined the universal covering space of the loop space of contractible loops in an almost contact metric manifold. Using the action functional, we study Floer type cohomologies on almost contact metric manifolds.

In section V we investigate the relation between quantum and Floer type cohomologies an almost contact metric manifolds with a closed fundamental 2-form.

Theorem I.1. Let an almost contact metric semipositive manifold $(M^{2n+1}, g, \varphi, \eta, \xi)$ be either cosymplectic, contact, or C-manifold. Then there are a quantum type cohomology $QH^*(M, \Lambda_\phi)$ and a Floer type cohomology $HF^*(M, \phi, H, \varphi)$ for a regular pair (H, φ), and they are naturally isomorphic

$$QH^*(M) \rightarrow HF^*(M, \phi, H, \varphi)$$

as modules over a Novikov ring Λ_ϕ.

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II. ALMOST CONTACT METRIC MANIFOLDS

Let M be a real (2n + 1)-dimensional smooth manifold. An almost co-complex structure on M is defined by a smooth (1, 1)- type tensor field φ, a smooth vector field ξ and a smooth 1-form η on M such that for each point $x \in M$,

$$\varphi_x^2 = -I + \eta_x \otimes \xi_x,$$
$$\eta_x(\xi_x) = 1,$$

where $I : T_xM \to T_xM$ is the identity map of the tangent space T_xM.

A Riemannian manifold M with a metric tensor g and with an almost co-complex structure (φ, ξ, η), such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), X, Y \in \Gamma(TM),$$

is called an almost contact metric manifold.

The fundamental 2-form ϕ of an almost contact metric manifold $(M, g, \varphi, \xi, \eta)$ is defined by

$$\phi(X, Y) = g(X, \varphi Y)$$

for all $X, Y \in \Gamma(TM)$. The (2n+1)-form $\phi^n \wedge \eta$ does not vanish on M, and so M is orientable.

The Nijenhuis tensor of the (1, 1)-type tensor φ is the (1, 2)-type tensor field N_φ defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - [X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

for all $X, Y \in \Gamma(TM)$, where $[X, Y]$ is the Lie bracket of X and Y. An almost cocomplex structure (φ, ξ, η) on M is said to be integrable if the tensor field $N_\varphi = 0$, and is normal if $N_\varphi + 2d\eta \otimes \xi = 0$. We follow the definitions and notations in [1, 16, 19].

Definition II.1. An almost contact metric manifold $(M, g, \varphi, \eta, \xi, \phi)$ is said to be
(1) cosymplectic (or almost co-Kähler) if $d\phi = 0$ and $d\eta = 0$,
(2) contact (or almost Sasakian) if $\phi = d\eta$,
(3) almost Kenmotsu if $d\eta = 0$ and

$$d\phi(X, Y, Z) = \frac{2}{3} \cdot S \sum_{X,Y,Z} \eta(x)\phi(Y, Z),$$

where $S\Sigma$ denotes the cyclic sum,
(4) an almost C-manifold if $d\phi = 0$, $d\eta \neq 0$, and $d\eta \neq \phi$,
(5) co-Kähler if M is an integrable cosymplectic manifold,
(6) Sasakian if M is a normal almost Sasakian manifold,
(7) Kenmotsu if M is a normal almost Kenmotsu manifold,
(8) a C-manifold if M is a normal almost C-manifold.

III. QUANTUM TYPE COHOMOLOGIES

Quantum cohomology is defined on symplectic manifolds. We studied quantum type cohomologies on cosymplectic, contact, and C-manifolds [8, 9, 11].

Let $(M, g, \varphi, \eta, \xi, \phi)$ be a (2n + 1)-dimensional cosymplectic manifold. Then in [11], we have:

(1) For a generic almost co-complex structure φ on M, the moduli space $\mathcal{M}(M; A, \varphi)$ of stable rational φ-coholomorphic maps, which represent a 2-dimensional homology class A, is a compact smooth stratified manifold with virtual dimension $2c_1(\mathcal{D})[A] + 2n$, where $\mathcal{D} = \{X \in TM \mid \eta(X) = 0\}$ is a distribution sub-bundle of TM .
(2) The quantum type cohomology $QH^*(M)$ of manifold M is an associative ring under the quantum type product, defined by Gromov-Witten type invariants.

Let $(M, g, \varphi, \eta, \xi, \phi)$ be a (2n+1)-dimensional contact manifold. Then in [8], we have:

(1) For a generic almost co-complex structure φ on M, the moduli space $\mathcal{M}(M; A, \varphi)$ of rational φ-coholomorphic maps, which represent $A \in H_2(M; \mathbb{Z})$, is a smooth manifold with virtual dimension $2c_1(\mathcal{D})[A] + 2n$.
(2) The quantum type cohomology $QH^*(M)$ of manifold M is an associative ring under a quantum type product.

Let $(M_i, g_i, \varphi_i, \eta_i, \xi_i, \phi_i)$, $i = 1, 2$, be the almost contact metric manifolds and (M, g, J, ω) be the product of M_1 and M_2. Then in [9], we have:

(1) If M_1 and M_2 are cosymplectic, then M is symplectic.
(2) If M_1 and M_2 are co-Kählerian, then M is Kählerian.
(3) If M_1 and M_2 are contact, then M is not symplectic.
(4) If M_1 is cosymplectic and M_2 is contact, then M is not symplectic.

Let $(M^{2n+1}, g, \varphi, \eta, \xi)$ be an almost contact metric manifold. Then the distribution $\mathcal{D} = \{X \in TM \mid \eta(X) = 0\}$ is an n-dimensional complex vector bundle on M.

Now fix the vector bundle $\mathcal{D} \to M$. As the symplectic manifolds, a (1, 1)-type tensor field $\varphi : \mathcal{D} \to \mathcal{D}$ with $\varphi^2 = -I$ is said to be tamed by ϕ if $\phi(X, \varphi X) > 0$ for $X \in \mathcal{D} \setminus \{0\}$ and is said to be compatible if $\phi(\varphi X, \varphi Y) = \phi(X, Y)$.

Assume that the almost contact metric manifold M has a closed fundamental 2-form ϕ, i.e., $d\phi = 0$. An almost contact metric manifold M with the ϕ is called semipositive if for every $A \in \pi_2(M)$, $\phi(A) > 0$, $c_1(\mathcal{D})[A] \geqslant 3 - n$, then $c_1(\mathcal{D})[A] > 0$.

We assume that an almost contact metric compact manifold $(M^{2n+1}, g, \varphi, \eta, \xi, \phi)$ is either cosymplectic or contact or almost C-manifold. Then the fundamental 2-form ϕ is closed. A smooth map $u : (\Sigma, j) \to (M, \varphi)$ from a Riemann surface (Σ, j) into the (M, φ) is said to be φ-coholomorphic if $du \circ j = \varphi \circ du$.

Let $\mathcal{D} := \{X \in TM \mid \eta(X) = 0\}$ be the distribution associated with the almost co-complex structure (φ, η, ξ) on M, then

$$\mathbb{C}^n \to \mathcal{D} \overset{\pi}{\to} M$$

is an n-dimensional complex vector bundle on M with an almost complex structure φ.

Lemma III.1. If $u : (\Sigma, j) \to (M, \varphi)$ is a φ-coholomorphic map from a Riemann surface (Σ, j) into the manifold (M, φ), then we have

(1) the image $\varphi(TM) \subset \mathcal{D}$ and $\varphi : \mathcal{D} \to \mathcal{D}$ with $\varphi^2 = -I$,
(2) the image $du(T\Sigma) \subset \mathcal{D}$,
(3) the diagram

$$\begin{array}{ccc} T\Sigma & \xrightarrow{du} & \mathcal{D} \\ \downarrow j & & \downarrow \varphi \\ T\Sigma & \xrightarrow{du} & \mathcal{D} \end{array}$$

is commutative, i.e., $\varphi \circ du = j \circ du$.

Let $A \in H_2(M; \mathbb{Z})$ be a 2-dimensional integral homology class in M. Let $\mathcal{M}_{0,3}(M; A, \varphi)$ be the moduli space of stable rational φ -coholomorphic maps with 3 marked points which represent class A.

Theorem III.2. For a generic almost complex structure φ on the distribution, $\mathbb{C}^n \to \mathcal{D} \to M$, the moduli space $\mathcal{M}_{0,3}(M; A, \varphi)$ is a compact stratified manifold with virtual dimension $2c_1(\mathcal{D})[A] + 2n$.

Consider the evaluation map given by

$$\mathrm{ev} : \mathcal{M}_{0,3}(M; A, \varphi) \to M^3,$$
$$\mathrm{ev}(\Sigma; z_1, z_2, z_3, u) = (u(z_1), u(z_2), u(z_3)).$$

We have Gromov-Witten type invariants given by

$$\Phi_{0,3}^{M,A,\varphi} : H^*(M^3) \to \mathbb{Q}$$
$$\Phi_{0,3}^{M,A,\varphi}(\alpha) = \int_{\mathcal{M}_{0,3}(M;A,\varphi)} \mathrm{ev}^*(\alpha)$$
$$= \mathrm{ev}_*[\mathcal{M}_{0,3}(M; A, \varphi)] \cdot \mathrm{PD}(\alpha)$$

to be the number of these intersection points counted with signs according to their orientations.

For two generic almost co-complex structures φ_0 and φ_1 connected by a generic smooth path φ_t in the space of almost co-complex structures on M, there is a cobodism $\mathcal{W}_{0,3}(M; A, \varphi_t)$ of dimension $2c_1(\mathcal{D})[A]+2n+1$ between $\mathcal{M}_{0,3}(M; A, \varphi_0)$ and $\mathcal{M}_{0,3}(M; A, \varphi_1)$. Thus, the Gromov-Witten type invariants are the same,

$$\Phi_{0,3}^{M,A,\varphi_0} = \Phi_{0,3}^{M,A,\varphi_1} : H^*(M^3) \to \mathbb{Q}.$$

We define a quantum type product $*$ on $H^*(M)$, for $\alpha \in H^k(M)$ and $\beta \in H^l(M)$,

$$\alpha * \beta = \sum_{A \in H_2(M)} (\alpha * \beta)_A q^{c_1(\mathcal{D})[A]/N},$$

where N is called the minimal Chern number defined by

$$< c_1(\mathcal{D}), H_2(M) > = N\mathbb{Z}.$$

The $(\alpha * \beta)_A \in H^{k+l-2c_1(\mathcal{D})[A]}(M)$ is defined by for each $C \in H_{k+l-2c_1(\mathcal{D})[A]}(M)$,

$$\int_C (\alpha * \beta)_A = \Phi_{0,3}^{M,A,\varphi}(\alpha \otimes \beta \otimes \gamma), \gamma = PD(C).$$

We define a quantum type cohomology of M by

$$QH^*(M) := H^*(M) \otimes \mathbb{Q}[q]$$

where $\mathbb{Q}[q]$ is the ring of Laurent polynomials in q with coefficients in the rational numbers $\mathbb{Q}$. By linearly extending the product $*$ on $QH^*(M)$, we have

Theorem III.3. The quantum type cohomology $QH^*(M)$ of the manifold M is an associative ring under the product $*$.

IV. Floer Type Cohomologies

In this section we assume that our manifold M is either a cosymplectic, contact, or C-manifold.

Let $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth 1-periodic family of Hamiltonian functions. Denoted by $X_t : M \to TM$ the Hamiltonian vector field defined by

$$\phi(X_t, \bullet) = dH_t(\bullet)$$

and consider the time dependent Hamiltonian differential equation

$$\dot{x}(t) = X_t(x(t)). \quad (1)$$

Let $(M^{2n+1}, g, \varphi, \eta, \xi, \phi)$ be an almost contact metric manifold. Since $dH_t(X_t) = \phi(X_t, X_t) = 0$ and $dH_t(\xi) = \phi(X_t, \xi) = g(X_t, \varphi\xi) = 0$, the vector fields ξ and X_t are in the kernel of dH_t, and so their integral curves lie in level sets of H_t.

The solutions of (1) generate a family of cosymplectomorphisms

$$\Psi_t : M \to M$$

with $\frac{d}{dt}\Psi_t = X_t \circ \Psi_t$, $\Psi_0 = id$. The fixed points of the time-1-map $\Psi = \Psi_1$ are one-to-one correspond with the 1-periodic solutions of (1).

Let $x : \mathbb{R}/\mathbb{Z} \to M$ be a contractible loop, then there is a smooth map $u : \mathcal{D} \to M$, defined on the unit disc $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, which satisfies $u(e^{2\pi i t}) = x(t)$. Two such maps $u_1, u_2 : D \to M$ are called equivalent if their boundary sum $u_1 \# (-u_2) : S^2 \to M$ is homologus to zero in the torsionless integral homology space $H_2(M)$/Tor.

Let $\tilde{x} := [x, u]$ be an equivalence class and denote by $\widetilde{LM}$ the space of equivalence classes. The space $\widetilde{LM}$ is the unique covering space of the space LM of contractible loops in M whose group of deck transformation is the image $H_2^S(M)$ of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$.

The space $H_2^S(M)$ acts on $\widetilde{LM}$ by

$$H_2^s(M) \times \widetilde{LM} \to \widetilde{LM} : (A, \tilde{x}) \to A\#\tilde{x}.$$

The symplectic type action functional $a_H : \widetilde{LM} \to \mathbb{R}$ is defined by

$$a_H([x,u]) = -\int_D u^*\phi - \int_0^1 H_t(x(t))dt,$$

then satisfies $a_H(A\#\tilde{x}) = a_H(\tilde{x}) - \phi(A)$.

Theorem IV.1. Let (M, ϕ) the manifold with a closed fundamental 2-form ϕ and fix a Hamiltonian function $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$. Let $[x, u] \in \widetilde{LM}$ and $V \in T_x\widetilde{LM} = C^\infty(\mathbb{R}/\mathbb{Z}, x^*TM)$. Then $(da_H)_{(x,[u])}(V) = \int_0^1 \phi(\dot{x} - X_{H_t}(x), V)dt$.

Proof. Choose two smooth maps:

$$\mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M, \quad (s,t) \mapsto x^s(t) = v(s,t)$$
$$\mathbb{R} \times D \to M, \quad (s,z) \mapsto u^s(z),$$

such that $x^0(t) = x(t)$, $u^0(z) = u(z)$, $u^s(e^{2\pi it}) = x^s(t)$, and $\frac{\partial}{\partial s}v(0,t) = V(t)$.

Then the map $\mathbb{R} \to \widetilde{LM}: s \mapsto (x^s, u^s)$ is the lift of the path $\mathbb{R} \to LM, s \mapsto x^s$ with $(x^0, u^0) = (x, u)$.

Since

$$a_H(x^s, u^s) = -\int_D u^{s^*}\phi - \int_0^1 H_t(v(s,t))dt$$
$$= -\int_0^s\int_0^1 \phi(\partial_s v, \partial_t v)dtds - \int_0^1 H_t(v(s,t))dt,$$

hence

$$\frac{d}{ds}a_H(x^s,u^s)\Big|_{s=0} = -\int_0^1 \phi\left(\partial_s v(s,t), \partial_t v - X_{H_t}(v(s,t))\right)\Big|_{s=0}$$
$$= \int_0^1 \phi(\dot{x} - X_{H_t}(x), V)dt.$$

□

Corollary IV.2. A point $[x, u] \in \widetilde{LM}$ is a critical point of the symplectic type action $a_H : \widetilde{LM} \to \mathbb{R}$ if and only if the contractible loop $x \in LM$ is a solution of (1). □

Let $(M^{2n+1}, g, \varphi, \eta, \xi)$ be a compact semipositive almost contact metric manifold with a closed fundamental 2-form ϕ. Let $\widetilde{P(H)}$ be the critical points of the symplectic type action functional $a_H : \widetilde{LM} \to \mathbb{R}$, and $\mu : \widetilde{P(H)} \to \mathbb{Z}$ a version of Maslov index defined in section III.

If $\mu(\tilde{x}) - \mu(\tilde{y}) = 1$, $\tilde{x}, \tilde{y} \in \widetilde{P(H)}$, then we denote

$$\eta(\tilde{x},\tilde{y}) := \#\left(\frac{\mathcal{M}(\tilde{x},\tilde{y})}{\mathbb{R}}\right),$$

where the connecting orbits are to be counted with signs determined by a system of coherent orientations of the moduli space $\mathcal{M}(\tilde{x},\tilde{y})$. These numbers give us a Floer type cochain complex. We fix the commutative ring R of real numbers. We introduce the Novikov ring Λ_ϕ of all functions $\lambda : H_2^s(M) \to \mathbb{R}$ that satisfy the finiteness condition $\#\{A \in H_2^s(M) \mid \lambda(A) = 0, \phi(A) \le c\} < \infty$ for all $c \in \mathbb{R}$.

Then Λ_ϕ is a $2\mathbb{Z}$-graded commutative ring and $\mathbb{R}$-module, where the grading is given by $\deg(A) = 2c_1(\mathcal{D})[A]$. Define the Floer cochain group $CF^* := CF^*(M, H)$ as the set of function $\xi : \widetilde{P(H)} \to \mathbb{R}$ that satisfy the finiteness condition $\#\{x \in \widetilde{P(H)} \mid \xi(x) = 0, a_H(x) \le c\} < \infty$ for all $c \in \mathbb{R}$. This complex CF^* is a module over Novikov ring Λ_ϕ with action given by

$$(\lambda * \xi)(\tilde{x}) := \sum_{A \in H_2^s(M)} \lambda(A)\xi(A\#\tilde{x}).$$

Here the grading is given by $\deg(\tilde{x}) = \mu(\tilde{x})$. The degree k part $CF^k(M, H)$ consists of all $\xi \in CF^*$ that are nonzero only on elements $\tilde{x} \in \widetilde{P(H)}$ with $\mu(\tilde{x}) = k$. Thus the action $\xi \mapsto \lambda * \xi$ changes the degree unless $\lambda(A)$ is nonzero only when $c_1(A) = 0$. The dimension of $CF^*(M, H)$ as a module over Λ_ϕ is the number $\#P(H)$ of contractible periodic solutions of (1).

Now we define a coboundary operator

$$\delta^k : CF^k(M,H) \to CF^{k+1}(M,H),$$
$$(\delta^k\xi)(\tilde{x}) = \sum_{\mu(\tilde{x})=\mu(\tilde{y})+1} \eta(\tilde{x},\tilde{y})\xi(\tilde{y})$$

where $\xi \in CF^k(M, H)$, $\mu(\tilde{x}) = k+1$ and $\mu(\tilde{y}) = k$.

Theorem IV.3. Let (M, φ) be a semipositive almost contact metric manifold with a closed fundamental 2-form. The coboundary operators defined as above $\delta^k : CF^k(M, H) \to CF^{k+1}(M, H)$ satisfy $\delta^{k+1} \circ \delta^k = 0$, for all k.

Let (M, φ) be a semipositive almost contact metric manifold with a closed fundamental 2-form ϕ. For a generic pair (H, φ) on M, the cochain complex (CF^*, δ) defines cohomology groups

$$HF^*(M, \phi, H, \varphi) := \frac{\ker \delta}{\operatorname{im} \delta}$$

which are called the Floer type cohomology groups of the (M, ϕ, H, φ). Since the coboundary map is linear over Λ_ϕ, the Floer type cohomology group $HF^*(M, \phi, H, \varphi)$ is a module over Novikov ring Λ_ϕ.

V. QUANTUM AND FLOER TYPE COHOMOLOGIES

In this section we assume that our manifold M is a compact either cosymplectic or contact or C-manifold. In section III we study a quantum type cohomology of M and in section IV a Floer type cohomology of M. We want to investigate relations between them on M. Let $(M^{2n+1}, g, \varphi, \eta, \xi)$ be a cosymplectic manifold. A smooth map $u : (\Sigma, j) \to (M, \varphi)$ from a Riemann surface (Σ, j) to the manifold (M, φ) is said to be φ-coholomorphic if $du \circ j = \varphi \circ du$.

In [8, 9, 11], we induced a quantum type cohomology ring $(QH^*(M), *)$ by using the moduli space of stable rational φ-coholomorphic maps, Gromov-Witten type invariants, and quantum type product $*$.

In [10] we studied a Floer type cohomology on cosymplectic manifolds M. The space LM of contractible loops in M has a natural universal covering space $\widetilde{LM}$ with fiber $H_2(M)$. The symplectic type action functional $a_H : \widetilde{LM} \to \mathbb{R}$ is defined by a smooth 1-periodic family of Hamiltonian type functions $H_t : M \to \mathbb{R}$. We have an infinite dimensional version of Morse-Novikov theory for the symplectic type action functional a_H. There are the gradient flow lines of a_H between critical points with indecies which are defined by Conley-Zehnder index.

The Floer type connecting orbits and the critical points of a_H yield a cochain complex of M. The cochain complex produces a Floer type cohomology of M.

Theorem V.1. Let $(M, g, \varphi, \eta, \xi)$ be a compact semipositive either cosymplectic or contact or C-manifold. Then, for every regular pair (H, φ), there is an isomorphism between Floer type cohomology and quantum type cohomology,

$$\Phi : HF^*(M, \phi, H, \varphi) \to QH^*(M, \Lambda_\phi).$$

Proof. Let $(M^{2n+1}, g, \varphi, \eta, \xi, \phi)$ be a compact semipositive either cosymplectic or contact or C-manifold. Let $f : M \to \mathbb{R}$ be a Morse function such that the negative gradient flow of f with respect to the metric $\phi(\,\bullet, \varphi(\bullet)) + \eta\otimes\eta$ is Morse-Smale and consider the time independent Hamiltonian $H_t := -\varepsilon f$, $t\in\mathbb{R}$.

If ε is sufficiently small, then the 1-periodic solutions of $\dot{x}(t) = X_t(x(t))$ are precisely the critical point of f. The Conley-Zehnder index

$$\mu(x, u_x) = n - \mathrm{ind}_f(x) = \mathrm{ind}_{-f}(x) - n$$

where $u_x : D^2 \to M$ is the constant map $u_x(z) = x$. For details, see [18].

The downward gradient flow lines $u : \mathbb{R} \to M$ of f are solutions of the ordinary differential equation

$$\dot{u}(s) + \nabla f(u(s)) = 0$$

and they form special solutions of the partial differential equation

$$\partial_s u + J(u)(\partial_t u - X_t(u)) = 0.$$

These solutions determine the Morse-Witten coboundary operator

$$\delta : C^*(M, f, \Lambda_\phi) \to C^*(M, f, \Lambda_\phi).$$

This coboundary operator is defined on the same cochain complex as the Floer coboundary δ and the cochain complex has the same grading for both complex $C^*(M, f, \Lambda_\phi)$ can be identified with the graded Λ_ϕ-module of all functions $\xi : Crit(f) \times H_2(M) \to \mathbb{R}$ that satisfy the finiteness condition $\#\{(x, A) \mid \xi(x, A) = 0, \phi(A)\geq c\} < \infty$ for all $c\in\mathbb{R}$.

The Λ_ϕ-module structure is given by

$$(\lambda * \xi)(x, A) = \sum_B \lambda(B)\xi(x, A + B),$$

the grading is $\deg(x, A) = \mathrm{ind}_f(x) - 2c_1(A)$, and the coboundary operator δ is defined by $(\delta\xi)(x, A) = \Sigma_y\, n_f(x, y)\xi(y, A)$, $(x, A) \in Crit(f)\times H_2^S(M)$, where $n_f(x, y)$ is the number of connecting orbits from x to y of shift equivalence classes of solutions of

$$\dot{u}(s) + \nabla f(u(s)) = 0,$$
$$lim_{s\to-\infty} u(s) = x,\ lim_{s\to+\infty} u(s) = y,$$

counted with appropriate signs.

Here we assume that the gradient flow of f is Morse-Smale and so the number of connecting orbits is finite when $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) = 1$.

With these conventions δ is a Λ_ϕ-module homomorphism of degree one and satisfies $\delta \circ \delta = 0$. It is called Morse-Witten type coboundary operator. Its cohomology is canonically isomorphic to the quantum type cohomology of M with coefficients in Λ_ϕ.

Now we define a chain map from the Floer type cochain complex to the Morse-Witten type cochain complex:

$$\Phi : CF^*(M, H) \to C^*(M, f, \Lambda_\phi),$$

for each $\tilde{x}\in\widetilde{P(H)}$ we denote $\mathcal{M}(\tilde{x}, H, \varphi)$ by the space of perturbed φ-coholomorphic maps $u : \mathbb{C} \to M$ such that $u(re^{2\pi i t})$ converges to a periodic solution $x(t)$ of the Hamiltonian system H_t as $r\to\infty$. The space has dimension $n - \mu(x)$. Now fix a Morse function $f : M \to \mathbb{R}$ such that the downward gradient flow $u : \mathbb{R} \to M$ satisfying

$$\dot{u}(s) + \nabla f(u(s)) = 0$$

is Morse-Smale. For a critical point $x\in Crit(f)$ the unstable manifold $W^u(x, f)$ of x has dimension $\mathrm{ind}_f(x)$ and codimension $2n - \mathrm{ind}_f(x)$ in the distribution $\mathcal{D}$. The submanifold of all $u\in\mathcal{M}(\tilde{x}, H, \varphi)$ with $u(0)\in W^u(x)$ has dimension $\mathrm{ind}_f(x) - \mu(\tilde{x}) - n = \dim\mathcal{M}(\tilde{x}, H, \varphi) + \dim W^u(x, f) - 2n$.

If $\mathrm{ind}_f(x) = \mu(\tilde{x}) + n$, then the space of spiked discs is 0-dimensional and hence the numbers $n(x, \tilde{x})$ of its elements can be used to construct the chain map

$$\Phi : CF^*(M, H) \to C^*(M, f, \Lambda_\phi)$$
$$(\Phi\xi)(x, A) := \sum_{\mathrm{ind}_f(x)=\mu(\tilde{x})+n} n(x, \tilde{x})\xi(A\#\tilde{x})$$

which is a Λ_ϕ-module homomorphism and raises the degree by n. The chain map Φ induces a homomorphism on cohomology. Thus we have

$$\Phi : HF^*(M, \Lambda_\phi) \to H^*(M, f, \Lambda_\phi) = \frac{\ker \delta}{\operatorname{im} \delta} \simeq QH^*(M, \Lambda_\phi),$$

since the Morse complex $C^*(M, f, \Lambda_\phi)$ and the Floer complex $CF^*(M, -\varepsilon f)$ are isomorphic as groups.

Similary, we can construct a chain map,

$$\Psi : C^*(M, f, \Lambda_\phi) \to CF^*(M, H),$$

$$(\Psi\xi)(\tilde{x}) = \sum_{\mu(\tilde{x})+n=\operatorname{ind}_f(y)-2c_1(A)} n((-A)\#\tilde{x}, y)\xi(y, A).$$

Then $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are chain homotopic to the identity. Thus we have an isomorphism Φ. □

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