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Numerical Solution of N^{TH} - Order Fuzzy Initial Value Problems by Fourth Order Runge-Kutta Method Basesd On **Contrahamonic Mean**

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Abstract--In this paper, a numerical method for N^{th} - order fuzzy initial value problems (FIVP) based on Seikkala derivative of fuzzy process is studied. The fourth order Runge-Kutta method based on Contra-harmonic Mean (RKCoM4) is used to find the numerical solution of this problem and the convergence and stability of the method is proved. This method is illustrated by solving second and third order FIVPs. The results show that the proposed method suits well to find the numerical solution of N^{th} – order FIVPs.

Keywords - *Fuzzy numbers, N^{th} - order Fuzzy Initial Value Problems, Runge-Kutta method, Contra-harmonic Mean, Lipschitz condition.*

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1. INTRODUCTION

The research work on Fuzzy Differential Equations (FDEs) has been rapidly developing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh[9], it was followed up by Dubois and Prade [10] by using the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [23] and Goetschel and Voxman [16]. Kandel and Byatt [21] applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [19, 20], Seikkala [24], He and Yi [17], and by other researchers [6, 8]. The numerical methods for solving fuzzy differential equations are introduced by Abbasbandy et.al. and Allahviranloo et.al. in [1, 2, 5]. Buckley and Feuring [7] introduced two analytical methods for solving N^{th} - order linear differential equations with fuzzy initial value conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, they first solved the fuzzy initial value problem and the checked to see if it defined a fuzzy function. Allahviranloo et.al [3, 4] proposed the methods for solving N^{th} – order fuzzy differential equations. Jayakumar et.al [18] used the Runge - Kutta Nystrom method for solving N^{th} – order fuzzy differential equations. Gethsi Sharmila and Henry Amirtharaj [14, 15] introduced the explicit third order Runge-Kutta method based on Centroidal Mean (CeM) to solve IVPs and developed a numerical algorithm for finding the solution of Fuzzy Initial Value Problems by Fourth Order Runge-Kutta Method Based on Contra-harmonic Mean.

In this paper, a numerical method to solve N^{th} - order linear fuzzy initial value problem is presented using the fourth order Runge – Kutta method based on Contra-harmonic Mean. The structure of the paper is organized as follows: In Section 2, some basic results on fuzzy numbers and fuzzy derivative are given. Then the fuzzy initial value problem is treated in Section 3 using the extension principle of Zadeh and the concept of fuzzy derivative. It is shown that the fuzzy initial value problem has a unique fuzzy solution when *f* satisfies Lipschitz condition which guarantees a unique solution to the deterministic initial value problem. In Section 4, the fourth order Runge-Kutta method based on Contra-harmonic Mean for solving N^{th} - order fuzzy initial value problems is introduced. In Section 5 convergence and stability are illustrated. In Section 6 the proposed method is illustrated by solving two examples, and the conclusion is drawn in Section 7.

2. PRELIMINARIES

An arbitrary fuzzy number is represented by an ordered pair of functions $\left(\underline{u}(r),\overline{u}(r)\right)$ for all $r \in [0,1]$, which satisfy the following requirements:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1] ,

(ii) $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1] ,

(iii) $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1.$

Let *E* be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $\alpha-$ level intervals.

Lemma 2.1

Let $\left[\underline{v}(\alpha),\overline{v}(\alpha)\right],\ \alpha\in(0,1]$ be a given family of non-empty intervals. If

$$(i)\ [\underline{v}(\alpha),\overline{v}(\alpha)]\supset[\underline{v}(\beta),\overline{v}(\beta)] \text{ for } 0<\alpha\le\beta, \text{ and}$$

$$(ii)\ [\lim_{k\to\infty}\underline{v}(\alpha_k),\lim_{k\to\infty}\overline{v}(\alpha_k)]=[\underline{v}(\alpha),\overline{v}(\alpha)],$$

whenever (α_k) is a non-decreasing sequence converging to $\alpha\in(0,1]$, then the family $[\underline{v}(\alpha),\overline{v}(\alpha)],\ \alpha\in(0,1]$, represent the $\alpha-$ level set of fuzzy number v in E. Conversely if $[\underline{v}(\alpha),\overline{v}(\alpha)],\ \alpha\in(0,1]$, are $\alpha-$ level set of fuzzy number $v\in E$ then the conditions (i) and (ii) hold true.

Definition 2.1

Let I be a real interval. A mapping $v:I\to E$ is called a fuzzy process and denote the $\alpha-$ level set by $[v(t)]_\alpha=[\underline{v}(t,\alpha),\overline{v}(t,\alpha)]$. The Seikkala derivative $v'(t)$ of v is defined by $[v'(t)]_\alpha=[\underline{v}'(t,\alpha),\overline{v}'(t,\alpha)]$, provided that is a equation defines a fuzzy number $v'(t)\in E$.

Definition 2.2

Suppose u and v are fuzzy sets in E. Then their Hausdorff

$$D:E\times E\to R_+\cup\{0\},\ D(u,v)=\sup_{\alpha\in[0,\,1]}\max\left\{\left|\underline{u}(\alpha)-\underline{v}(\alpha)\right|,\left|\overline{u}(\alpha)-\overline{v}(\alpha)\right|\right\},$$

i.e $D(u,v)$ is maximal distance between α level sets of u and v.

3. FUZZY INITIAL VALUE PROBLEM

Now we consider the initial value problem

$$\left\{x^{(n)}(t)=\psi(t,x,x',\dots,x^{(n-1)}),\ x(0)=a_1,\dots,\ x^{(n-1)}(0)=a_n,\right. \tag{3.1}$$

where ψ is a continuous mapping from $R_+\times R^n$ into R and $a_i(0\le i\le n)$ are fuzzy numbers in E. The mentioned N^{th} - order fuzzy differential equation by changing variables

$$y_1(t)=x(t),\ y_2(t)=x'(t),\dots,\ y_n(t)=x^{(n-1)}(t),$$

converts to the following fuzzy system

$$\begin{cases} y_1'(t)=f_1(t,y_1,\dots,y_n),\\ \vdots\\ y_n'(t)=f_n(t,y_1,\dots,y_n),\\ y_1(0)=y_1^{[0]}=a_1,\dots,\ y_n(0)=y_n^{[0]}=a_n,\end{cases} \tag{3.2}$$

where $f_i\ (1\le i\le n)$ are continuous mapping from $R_+\times R^n$ into R and $y_i^{[0]}$ are fuzzy numbers in E with α - level intervals.

$$[y_i^{[0]}]_\alpha=[\underline{y}_i^{[0]}(\alpha),\overline{y}_i^{[0]}(\alpha)] \text{ for i = 1,..., n and } 0\le\alpha\le1$$

We call $y=(y_1,\dots,y_n)^T$ is a fuzzy solution of (3.2) on an interval I, if

$$\begin{aligned}\underline{y}'_i(t,\alpha)&=\min\left\{f_i(t,u_1,\dots,u_n);\ u_j\in[\underline{y}_j(t,\alpha),\ \overline{y}_j(t,\alpha)]\right\}=\underline{f}_i(t,y(t,\alpha)),\\ \overline{y}_i'(t,\alpha)&=\max\left\{f_i(t,u_1,\dots,u_n);\ u_j\in[\underline{y}_j(t,\alpha),\ \overline{y}_j(t,\alpha)]\right\}=\overline{f}_i(t,y(t,\alpha)),\end{aligned} \tag{3.3}$$

and

$$\underline{y}_i(0,\alpha)=\underline{y}_i^{[0]}(\alpha),\ \overline{y}_i(0,\alpha)=\overline{y}_i^{[0]}(\alpha) \tag{3.4}$$

Thus for fixed α we have a system of initial value problem in R^{2n}. If we can solve it (uniquely), we have only to verify that the intervals, $[\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)]$ define a fuzzy number $y_i(t) \in E$. Now let $\underline{y}^{[0]}(\alpha) = (\underline{y}_1^{[0]}(\alpha), ..., \underline{y}_n^{[0]}(\alpha))^T$ and $\overline{y}^{[0]}(\alpha) = (\overline{y}_1^{[0]}(\alpha), ..., \overline{y}_n^{[0]}(\alpha))^T$ with respect to the above mentioned indicators, system (3.2) can be written as with assumption

$$\begin{cases} y'(t) = F(t, y(t)), \\ y(0) = y^{[0]} \in E^n. \end{cases} \tag{3.5}$$

With assumption

$$y(t,\alpha) = [\underline{y}(t,\alpha), \overline{y}(t,\alpha)] \text{ and } y'(t,\alpha) = [\underline{y}'(t,\alpha), \overline{y}'(t,\alpha)]$$

where

$$\underline{y}(t,\alpha) = [\underline{y}(t,\alpha), ..., \underline{y}(t,\alpha)]^T, \tag{3.6}$$

$$\overline{y}(t,\alpha) = [\overline{y}(t,\alpha), ..., \overline{y}(t,\alpha)]^T, \tag{3.7}$$

$$\underline{y}'(t,\alpha) = [\underline{y}'(t,\alpha), ..., \underline{y}'(t,\alpha)]^T, \tag{3.8}$$

$$\overline{y}'(t,\alpha) = [\overline{y}'(t,\alpha), ..., \overline{y}'(t,\alpha)]^T, \tag{3.9}$$

and with assumption $F(t, y(t,\alpha)) = [\underline{F}(t, y(t,\alpha))), \overline{F}(t, y(t,\alpha))]$, where

$$\underline{F}(t, y(t,\alpha)) = [\underline{f}_1(t, y(t,\alpha))), ..., \underline{f}_n(t, y(t,\alpha))]^T, \tag{3.10}$$

$$\overline{F}(t, y(t,\alpha)) = [\overline{f}_1(t, y(t,\alpha))), ..., \overline{f}_n(t, y(t,\alpha))]^T, \tag{3.11}$$

y (t) is a fuzzy solution of (3.5) on an interval *I* for all $\alpha \in (0,1]$, if

$$\begin{cases} \underline{y}'(t,\alpha)) = \underline{F}(t, y(t,\alpha)); \\ \overline{y}'(t,\alpha)) = \overline{F}(t, y(t,\alpha)) \\ \underline{y}(0,\alpha) = \underline{y}^{[0]}(\alpha), \ \overline{y}(0,\alpha) = \overline{y}^{[0]}(\alpha) \end{cases} \tag{3.12}$$

or

$$\begin{cases} y'(t,\alpha)) = F(t, y(t,\alpha)), \\ y(0,\alpha) = y^{[0]}(\alpha). \end{cases} \tag{3.13}$$

Now we show that under the assumptions for functions $f_{i,}$ *for* i=1,..., n how we can obtain a unique fuzzy solution for system (3.2).

Theorem 3.1

If $f_i(t, u_1, ..., u_n)$ for $i = 1, ..., n$ are continuous function of *t* and satisfies the Lipschitz condition in $u = (u_1, ..., u_n)^T$ in the region $D = \{t, u | t \in I = [0,1], -\infty < u_i < \infty \text{ for } i=1,..., n\}$ with constant L_i then the initial value problem (3.2) has a unique fuzzy solution in each case.

Proof. Denote $G = (\underline{F}, \overline{F})^T = (\underline{f}_1, ..., \underline{f}_n, \overline{f}_1, ..., \overline{f}_n)^T$ where

$$\underline{f}_i(t,u) = \min\{ f_i(t, u_1, ..., u_n);\ u_j \in [\underline{y}_j, \overline{y}_j],\ for\ j = 1, ..., n\}, \tag{3.14}$$

$$\overline{f}_i(t,u) = \max\{ f_i(t, u_1, ..., u_n);\ u_j \in [\underline{y}_j, \overline{y}_j],\ for\ j = 1, ..., n\}, \tag{3.15}$$

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Denote $y=(\underline{y},\overline{y})^T=(\underline{y}_1,\dots,\ \underline{y}_n,\overline{y}_n,\dots,\ \overline{y}_n)^T\in\ \mathrm{R}^{2n}$. It can be shown that Lipschitz condition of functions f_i imply

$$\left\|F(t,z)-F(t,z^{\bullet})\right\|\le L\left\|z-z^{\bullet}\right\|$$

This guarantees the existence and uniqueness solution of

$$\begin{cases} y'(t))=F(t,y(t)), \\ y(0)=y^{[0]}=(\underline{y}^{[0]},\overline{y}^{[0]})^{2t}\in R^{2n} \end{cases} \tag{3.16}$$

Also for any continuous function $y^{[1]}:R_+\to R^{2n}$ the successive approximations

$$y^{[m+1]}(t)=y^{[0]}+\int_0^t F(s,y^{[m]}(s))ds,\ \mathrm{t}\ge 0,\ \mathrm{m}=1,2,... \tag{3.17}$$

converge uniformly on closed subintervals of □ R_+ to the solution of (3.16). In other word we have the following successive approximations

$$\underline{y_i}^{[m+1]}(t)=\underline{y_i}^{[0]}+\int_o^t \underline{f}_i(s,y^{[m]}(s))ds,\ \text{for i=1,..., n,} \tag{3.18}$$

$$\overline{y_i}^{[m+1]}(t)=\overline{y_i}^{[0]}+\int_o^t \overline{f}_i(s,y^{[m]}(s))ds,\ \text{for i=1,..., n.} \tag{3.19}$$

By choosing $y^{[0]}=(\underline{y}^{[0]}(\alpha),\overline{y}^{[0]}(\alpha))$ in (3.16) we get a unique solution

$y^{\alpha}(t)=(\underline{y}(t,\alpha),\ \overline{y}(t,\alpha))$ to (3.3) and (3.4) for each $\alpha\in(0,1]$.

Next we will show that the $y(t,\alpha)=(\underline{y}(t,\alpha),\overline{y}(t,\alpha)),$ defines a fuzzy number in E^n for each $0\le t\le T,$ i.e. that $y=(y_1,\dots,\ \mathrm{y}_n)^T$ is a fuzzy solution to (3.14) and (3.15). Thus we will show that the intervals $[\underline{y}_i(t,\alpha),\overline{y}_i(t,\alpha)],$ for $i=1,\dots,\ \mathrm{n}$ satisfy the conditions of Lemma (2.1). The successive approximations $y^{[m]}=y^{[0]}\in E^n,\ y^{[m+1]}(t)=y^{[0]}+\int_o^t F(s,y^{[m]}(s))ds,\ \mathrm{t}\ge 0,\ \mathrm{m}=1,2,...,$

(3.20)

where the integrals are the fuzzy integrals, define a sequence of fuzzy numbers $y^{[m]}(t)=(y_1^{[m]}(t),\dots,\ y_n^{[m]}(t))^T$ for each $0\le t\le T.$ Hence $[y_i^{[m]}(t)]_\beta\subset(y_i^{[m]}(t)]_\alpha,$ if $\mathrm{o}<\alpha\le\beta\le 1,$ which implies that $[\underline{y}_i^{[m]}(t,\beta),\ \overline{y}_i^{[m]}(t,\beta)]\subset[\underline{y}_i^{[m]}(t,\alpha),\ \overline{y}_i^{[m]}(t,\alpha)],\ (0<\alpha\le\beta\le 1),$ since, by the convergence of sequences (3.16) and (3.19) , the end points of $[y_i^{[m]}(t)]_\alpha$ converge to $\underline{y}_i(t,\alpha)\ and\ \overline{y}_i(t,\alpha)$ that means

$\underline{y}_i^{[m]}(t,\alpha)\to\underline{y}_i(t,\alpha)$ and $\overline{y}_i^{[m]}(t,\alpha)\to\overline{y}_i(t,\alpha).$ (3.21)

Thus the inclusion property (i) of Lemma (2.1) holds for the intervals $[\underline{y}_i(t,\alpha),\overline{y}_i(t,\alpha)]$ for $0<\alpha\le 1.$ For the proof of the property (ii) of Lemma (2.1), let (α_p) be a non-decreasing sequence in (0,1] converging to α. Then $\underline{y}^{[0]}(\alpha_p)\to\underline{y}^{[0]}(\alpha)$ and $\overline{y}^{[0]}(\alpha_p)\to\overline{y}^{[0]}(\alpha)$, because of $\underline{y}^{[0]}\in E^n$. But by the continuous dependence on the initial value of the solution (3.16), $\underline{y}(t,\alpha_p)\to\underline{y}(t,\alpha)$ and $\overline{y}(\alpha_p)\to\overline{y}(\alpha),$ this means (ii) holds for the intervals $[\underline{y}(t,\alpha),\overline{y}(t,\alpha)],$ for $0<\alpha\le 1.$ Hence by Lemma (2.1), $y(t)\in E^n$ and so ***y*** is a fuzzy solution of (3.1). The uniqueness follows from the uniqueness of the solution of (3.16).

4. THE FOURTH ORDER RUNGE - KUTTA METHOD BASED ON CONTRAHARMONIC MEAN (RKCOM) TO SOLVE FUZZY INITIAL VALUE PROBLEMS

4.1. RKCoM formula for Solving system of IVPs

Evans and Yaakub[11] have developed a new RK method of order 4 based on Contra-harmonic Mean (RKCoM) to solve first order equation and it is to be noted that the Contra-harmonic Mean of two points x_1 and x_2 is defined as $\left(\frac{x_1^2+x_2^2}{x_1+x_2}\right)$ Murugesan et.al. [22] extended the fourth order RK formula based on Contra-harmonic Mean to solve system of IVPs (3.2) as follows:

$$y_{n+1j} = y_{nj} + \frac{h}{3}\left[\frac{k_{1j}^2+k_{2j}^2}{k_{1j}+k_{2j}} + \frac{k_{2j}^2+k_{3j}^2}{k_{2j}+k_{3j}} + \frac{k_{3j}^2+k_{4j}^2}{k_{3j}+k_{4j}}\right], \quad 1 \le j \le m$$

where

$$k_{1j} = f_j\left(x_n, y_{n1}, y_{n2}, \dots, y_{nm}\right)$$

$$k_{2j} = f_j\left(x_n + a_1 h, y_{n1} + a_1 h k_{11}, y_{n2} + a_1 h k_{12}, \dots, y_{nm} + a_1 h k_{1m}\right)$$

$$k_{3j} = f_j\left(x_n + (a_2+a_3)h, y_{n1} + a_2 h k_{11} + a_3 h k_{21}, y_{n2} + a_2 h k_{12} + a_3 h k_{22}, \dots, y_{nm} + a_2 h k_{1m} + a_3 h k_{2m}\right)$$

$$k_{4j} = f_j\begin{pmatrix} x_n + (a_4+a_5+a_6)h, y_{n1} + a_4 h k_{11} + a_5 h k_{21} + a_6 h k_{31}, y_{n2} + a_4 h k_{12} + a_5 h k_{22} + a_6 h k_{32}, \dots, \\ y_{nm} + a_4 h k_{1m} + a_5 h k_{2m} + a_6 h k_{3m} \end{pmatrix}$$

The parameters are:

$$a_1 = \frac{1}{2},\ a_2 = \frac{1}{8},\ a_3 = \frac{3}{8},\ a_4 = \frac{1}{4},\ a_5 = \frac{-3}{4},\ a_6 = \frac{3}{2} \quad (4.2)$$

4.2. Procedure for Solving FIVPs

We consider fuzzy initial value problem (3.2) with the unique solution $y = (y_{1,}, \dots,\ y_n)^T \in E^n$. For finding an approximate solution of (3.2) with the fourth order Runge- Kutta method based on Contra-harmonic Mean, we first define

$$\begin{aligned} \underline{y}(t_{n+1};\ r) - \underline{y}(t_n;\ r) &= \sum_{i=1}^{4} w_i \underline{k}_i(t_n, y(t_n;\ r), h), \\ \overline{y}(t_{n+1};\ r) - \overline{y}(t_n;\ r) &= \sum_{i=1}^{4} w_i \overline{k}_i(t_n,\ y(t_n;\ r), h), \end{aligned} \quad (4.3)$$

where the w_i's are constants and

$$[k_i(t,\ y(t;\ r), h)]_r = [\underline{k}_i(t,\ y(t;\ r), h),\ \overline{k}_i(t,\ y(t;\ r), h)],\ i = 1,\ 2,\ 3,\ 4$$

$$\begin{aligned} \underline{k}_i(t_n, y(t_n;\ r), h) &= f(t_n + c_i h,\ \underline{y}(t_n) + \sum_{j=1}^{i-1} a_{ij}\ \underline{k}_j(t_n,\ y(t_n;\ r), h)), \\ \overline{k}_i(t_n,\ y(t_n;\ r), h) &= f(t_n + c_i h,\ \overline{y}(t_n) + \sum_{j=1}^{i-1} a_{ij}\ \overline{k}_j(t_n,\ y(t_n;\ r), h)), \end{aligned} \quad (4.4)$$

$$\underline{k}_{i1}(t,\ y(t;r), h) = \min\{ f_i(t,\ \mathrm{s}_1, \dots,\ \mathrm{s}_n) \backslash s_j \in [\underline{y}_j(t;r),\ \overline{y}_j(t;r)]\},\ (1 \le \mathrm{i, j} \le \mathrm{n})$$

$$\overline{k}_{i1}(t,\ y(t;r), h) = \max\{ f_i(t,\ \mathrm{s}_1, \dots,\ \mathrm{s}_n) \backslash s_j \in [\underline{y}_j(t;r),\ \overline{y}_j(t;r)]\},$$

$$\underline{k}_{i2}(t,\ y(t;r),h)=\min\{\ f_i(t+\frac{h}{2},\ s_1,...,\ s_n)\setminus s_j\in[\underline{z}_{j1}(t;r),\ \overline{z}_{j1}(t;r)]\}$$
$$\overline{k}_{i2}(t,\ y(t;r),h)=\max\{\ f_i(t+\frac{h}{2},\ s_1,...,\ s_n)\setminus s_j\in[\underline{z}_{j1}(t;r),\ \overline{z}_{j1}(t;r)]\}$$
$$\underline{k}_{i3}(t,\ y(t;r),h)=\min\{\ f_i(t+\frac{h}{2},\ s_1,...,\ s_n)\setminus s_j\in[\underline{z}_{j2}(t;r),\ \overline{z}_{j2}(t;r)]\}$$
$$\overline{k}_{i3}(t,\ y(t;r),h)=\max\{\ f_i(t+\frac{h}{2},\ s_1,...,\ s_n)\setminus s_j\in[\underline{z}_{j2}(t;r),\ \overline{z}_{j2}(t;r)]\}$$
$$\underline{k}_{i4}(t,\ y(t;r),h)=\min\{\ f_i(t+h,\ s_1,...,\ s_n)\setminus s_j\in[\underline{z}_{j3}(t;r),\ \overline{z}_{j3}(t;r)]\}$$
$$\overline{k}_{i4}(t,\ y(t;r),h)=\max\{\ f_i(t+h,\ s_1,...,\ s_n)\setminus s_j\in[\underline{z}_{j3}(t;r),\ \overline{z}_{j3}(t;r)]\} \qquad (4.5)$$

such that ,

$$\underline{z}_{j1}(t,\ y(t;r),h)=\underline{y}_j(t;r)+\frac{h}{2}\underline{k}_{j1}(t,\ y(t;r))$$
$$\overline{z}_{j1}(t,\ y(t;r),h)=\overline{y}_j(t;r)+\frac{h}{2}\overline{k}_{j1}(t,\ y(t;r))$$
$$\underline{z}_{j2}(t,\ y(t;r),h)=\underline{y}_j(t;r)+\frac{h}{8}\underline{k}_{j1}(t,\ y(t;r),h)+\frac{3}{8}h\underline{k}_{j2}(t,y(t;r),h)$$
$$\overline{z}_{j2}(t,\ y(t;r),h)=\overline{y}_j(t;r)+\frac{h}{8}\overline{k}_{j1}(t,\ y(t;r),h)+\frac{3}{8}h\overline{k}_{j2}(t,y(t;r),h)$$
$$\underline{z}_{j3}(t,\ y(t;r),h)=\underline{y}_j(t;r)+\frac{h}{4}\underline{k}_{j2}(t,\ y(t;r),h)-\frac{3}{4}h\underline{k}_{j2}(t,y(t;r),h+\frac{3}{2}h\underline{k}_{j3}(t,y(t;r))$$
$$\overline{z}_{j3}(t,\ y(t;r),h)=\overline{y}_j(t;r)+\frac{h}{4}\overline{k}_{j2}(t,\ y(t;r),h)-\frac{3}{4}h\overline{k}_{j2}(t,y(t;r),h+\frac{3}{2}h\overline{k}_{j3}(t,y(t;r))$$

now we consider the following relations

$$F_i\left[t,\ y(t;r),h\right]=\frac{\underline{k}_{i1}{}^2(t,\ y(t;r),h)+\underline{k}_{i2}^2(t,\ y(t;r),h)}{\underline{k}_{i1}(t,\ y(t;r),h)+\underline{k}_{i2}(t,\ y(t;r),h)}+\frac{\underline{k}_{i2}^2(t,\ y(t;r),h)+\underline{k}_{i3}^2(t,\ y(t;r),h)}{\underline{k}_{i2}(t,\ y(t;r),h)+\underline{k}_{i3}(t,\ y(t;r),h)}+\frac{\underline{k}_{i3}^2(t,\ y(t;r),h)+\underline{k}_{i4}^2(t,\ y(t;r),h)}{\underline{k}_{i3}(t,\ y(t;r),h)+\underline{k}_{i4}(t,\ y(t;r),h)}$$

$$G_i\left[t,\ y(t;r),h\right]=\frac{\overline{k}_{i1}{}^2(t,\ y(t;r),h)+\overline{k}_{i2}^2(t,\ y(t;r),h)}{\overline{k}_{i1}(t,\ y(t;r),h)+\overline{k}_{i2}(t,\ y(t;r),h)}+\frac{\overline{k}_{i2}^2(t,\ y(t;r),h)+\overline{k}_{i3}^2(t,\ y(t;r),h)}{\overline{k}_{i2}(t,\ y(t;r),h)+\overline{k}_{i3}(t,\ y(t;r),h)}+\frac{\overline{k}_{i3}^2(t,\ y(t;r),h)+\overline{k}_{i4}^2(t,\ y(t;r),h)}{\overline{k}_{i3}(t,\ y(t;r),h)+\overline{k}_{i4}(t,\ y(t;r),h)}$$

and suppose that the discrete equally spaced grid points $\{t_0=0,t_1,...,\ t_N=T\}$ is a partition for interval [0, *T*]. If the exact and approximate solution in the *i*-th α cut at $t_m, 0\le m\le N$ are denoted by $[\underline{y}_i^{[m]}(\alpha),\overline{y}_i^{[m]}(\alpha)]$ and $[\underline{w}_i^{[m]}(\alpha),\overline{w}_i^{[m]}(\alpha)]$ respectively, then the numerical method for solution approximation in the *i*-th coordinate α cut, with the Runge-Kutta method based on contra-harmonic Mean is

$$\underline{w}_i^{[m+1]}(\alpha)=\underline{w}_i^{[m]}(\alpha)+\frac{h}{3}\underline{F}_i(t_m,w^m(\alpha),h),\quad \underline{w}_i^{[0]}(\alpha)=\underline{y}_i^{[0]}(\alpha),$$
$$\overline{w}_i^{[m+1]}(\alpha)=\overline{w}_i^{[m]}(\alpha)+\frac{h}{3}\overline{F}_i(t_m,w^m(\alpha),h),\quad \overline{w}_i^{[0]}(\alpha)=\overline{y}_i^{[0]}(\alpha),$$

Where

$$[w_i(t)]_\alpha = [\underline{w}_i(t,\alpha), \overline{w}_i(t,\alpha)],\ \mathrm{w}^{[m]}(\alpha) = [\underline{w}^{[m]}(\alpha)\overline{w}^{[m]}(\alpha)]$$
$$\underline{\mathrm{w}}^{[m]}(\alpha) = (\underline{w}_i^{[m]}(\alpha),\dots,\ \underline{w}_n^{[m]}(\alpha))^T, \text{and}\ \overline{\mathrm{w}}^{[m]}(\alpha) = (\overline{w}_i^{[m]}(\alpha),\dots,\ \overline{w}_n^{[m]}(\alpha))^T$$

Now we input

$$F^*(t, w^{[m]}(\alpha), h) = \frac{1}{3}(F_1(t, w^{[m]}(\alpha), h),\dots,\ F_n(t, w^{[m]}(\alpha), h)^T, \tag{4.22}$$
$$G^*(t, w^{[m]}(\alpha), h) = \frac{1}{3}(G_1(t, w^{[m]}(\alpha), h),\dots,\ \mathrm{G}_n(t, w^{[m]}(\alpha), h)^T. \tag{4.23}$$

The Runge-Kutta method based on Contra-harmonic Mean for solutions approximation α -cut of differential equation (3.13) is as follow

$$w^{[m+1]}(\alpha) = w^{[m]}(\alpha) + hH(t_m, w^m(\alpha), h),\ \ \mathrm{w}^{[0]}(\alpha) = y^{[0]}(\alpha) \tag{4.24}$$

where

$$H(t_m, w^{[m]}(\alpha), h) = F^*(t, w^{[m]}(\alpha), h) = [F^*(t_m, w^{[m]}(\alpha), h),\dots,\ \mathrm{G}^*(t_m, w^{[m]}(\alpha), h)]$$

and

$$F^*(t, w^{[m]}(\alpha), h) = \frac{1}{3}[\frac{\underline{k}_1^2(t,\ y(t;r),h) + \underline{k}_2^2(t,\ y(t;r),h)}{\underline{k}_1(t,\ y(t;r),h) + \underline{k}_2(t,\ y(t;r),h)} + \frac{\underline{k}_2^2(t,\ y(t;r),h) + \underline{k}_3^2(t,\ y(t;r),h)}{\underline{k}_2(t,\ y(t;r),h) + \underline{k}_3(t,\ y(t;r),h)} + \frac{\underline{k}_3^2(t,\ y(t;r),h) + \underline{k}_4^2(t,\ y(t;r),h)}{\underline{k}_3(t,\ y(t;r),h) + \underline{k}_4(t,\ y(t;r),h)}] \tag{4.25}$$

$$\mathrm{G}^*(t_m, w^{[m]}(\alpha), h)] = \frac{1}{3}[\frac{\overline{k}_1^2(t,\ y(t;r),h) + \overline{k}_2^2,(t,\ y(t;r),h)}{\overline{k}_1(t,\ y(t;r),h) + \overline{k}_2(t,\ y(t;r),h)} + \frac{\overline{k}_2^2(t,\ y(t;r),h) + \overline{k_3^2}(t,\ y(t;r),h)}{\overline{k}_2(t,\ y(t;r),h) + \overline{k}_3(t,\ y(t;r),h)} + \frac{\overline{k}_3^2(t,\ y(t;r),h) + \overline{k}_4^2(t,\ y(t;r),h)}{\overline{k}_3(t,\ y(t;r),h) + \overline{k}_4(t,\ y(t;r),h)}] \tag{4.26}$$

and also

$$\underline{k}_j(t, w^{[m]}(\alpha), h) = (\underline{k}_{1j}(t, w^{[m]}(\alpha), h),\dots,\ \underline{k}_{nj}(t, w^{[m]}(\alpha), h))^T,$$
$$\overline{k}_j(t, w^{[m]}(\alpha), h) = (\overline{k}_{1j}(t, w^{[m]}(\alpha), h),\dots,\ \overline{k}_{nj}(t, w^{[m]}(\alpha), h))^T. \qquad \text{(j=1,2,3,4)}$$

5. CONVERGENCE AND STABILITY

Definition 5.1

A one-step method for approximating the solution of a differential equation

$$\begin{cases} y'(t)) = F(t, y(t)), \\ y(0) = y^{[0]} \in R^n \end{cases} \tag{5.27}$$

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which F is a N[th] - ordered as follow $f=(f_1,\ldots, f_n)^T$ and $f_1: \mathrm{R}_+\times R^n \to R(1\le i\le n)$, is a method which can be written in the form

$$w^{[n+1]}=w^{[n]}+h\psi\ (t_n, w^n, h), \tag{5.28}$$

where the increment function ψ is determined by F and is a function of $t_n, w^{[n]}$ and h only.

Theorem 5.1

If $\psi\ (t,y,h)$ satisfies a Lipschitz condition in y, then the method given by (5.28) is stable.

Theorem 5.2

In relation (3.5), if $F(t, y)$ satisfies a Lipschitz condition in y, then the method given by (4.24) is stable.

Theorem 5.3

$$\text{If}\quad w^{[m+1]}(\alpha)=w^{[m]}(\alpha)+h\ \psi\ (t_m, w^m(\alpha),h),\quad \mathrm{w}^{[0]}(\alpha)=y^{[m]}(\alpha) \tag{5.29}$$

where $\psi\ (t_m, w^m(\alpha),h)=[\psi_1\ (t_m, w^m(\alpha),h),\psi_2\ (t_m, w^m(\alpha),h)]$ is a numerical method for approximation of differential equation (3.13), and ψ_1 and ψ_2 are continuous in t, y, h for $0\le t\le T,\ 0\le h\le h_0$ and all y, and if they satisfy a Lipschitz condition in the region $D=\{t,u,v,h)|0\le t\le T,\ -\infty<u_i\le v_i,\ -\infty<v_i\le+\infty,\ 0\le h\le h_0\ \text{i=1,..., n}\}$, necessary and sufficient conditions for convergence above mentioned method is $\psi(t,y(t,\alpha),h)=F(t,y(t,\alpha))$.

(5.30)

Proof: Suppose that $\psi(t,y(t,\alpha),0)=F(t,y(t,\alpha))$, since, $F(t,y(t,\alpha))$ satisfying the conditions of theory (3.1), then the following equation

$$\begin{cases} y'(t))=F(t,y(t)),\\ y(0)=y^{[0]}(\alpha)\end{cases} \tag{5.31}$$

has a unique solution such as $y(t,\alpha)=(\underline{y}(t,\alpha),\overline{y}(t,\alpha))$, where

$\underline{y}(t,\alpha)=(\underline{y}(t,\alpha),\ldots,\ \underline{y}(t,\alpha))^T$ and $\overline{y}(t,\alpha)=(\overline{y}(t,\alpha),\ldots,\ \overline{y}(t,\alpha))^T$. We will show that the numerical solutions given by (5.29) convergent to the $y(t)$. By the mean value theorem,

$$\underline{y_i}^{[m+1]}=\underline{y_i}^{[m]}+h\,\underline{f_i}(t_m+\underline{\theta}_i h, y(t_m+\underline{\theta}_i h)),\ \text{for } 0<\underline{\theta}_i<1, \tag{5.32}$$

$$\overline{y_i}^{[m+1]}=\overline{y_i}^{[m]}+h\,\overline{f_i}(t_m+\overline{\theta}_i h, y(t_m+\overline{\theta}_i h)),\ \text{for } 0<\overline{\theta}_i<1 \tag{5.33}$$

with assumption $\underline{\psi}=(\underline{\psi}_1,\ldots,\ \underline{\psi}_n)^T$ and $\overline{\psi}=(\overline{\psi}_1,\ldots,\ \overline{\psi}_n)^T$. From equation (5.29) obtain the following relations

$$\underline{w_i}^{[m+1]}(\alpha)=\underline{w_i}^{[m]}(\alpha)+h\psi_i(t_m, w^m(\alpha),h),$$

$\overline{w_i}^{[m+1]}(\alpha)=\overline{w_i}^{[m]}(\alpha)+h\psi_i(t_m, w^m(\alpha),h)$, and subtracting (5.32) , (5.33) from (5.34) , (5.35) respectively, and setting $e^{[m]}(\alpha)=[\underline{e}^{[m]}(\alpha),\overline{e}^{[m]}(\alpha)]$, where

$\underline{e}^{[m]}(\alpha)=\underline{e}(t_m,\alpha)=\underline{w}^{[m]}(\alpha)-\underline{y}^{[m]}(\alpha)$ and $\overline{e}^{[m]}(\alpha)=\overline{e}(t_m,\alpha)=\overline{w}^{[m]}(\alpha)-\overline{y}^{[m]}(\alpha)$, we get

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$$\underline{e}^{[m+1]}(\alpha) = \underline{e}_i^{[m]}(\alpha) + h\{\underline{\psi_i}(t_m, w^{[m]}(\alpha), h) - \underline{\psi_i}(t_m, w^{[m]}(\alpha), h) + \underline{\psi_i}(t_m, w^{[m]}(\alpha), h) - \underline{\psi_i}(t_m, w^{[m]}(\alpha), 0) + \underline{\psi_i}(t_m, w^{[m]}(\alpha), 0) - \underline{f}_i(t_m + \underline{\theta}_i h, y(t_m + \underline{\theta}_i h))\}$$

on the other way, with respect to the relation of $\underline{\psi_i}(t_m, w^{[m]}(\alpha), 0) - \underline{f}_i(t_m, y^{[m]}(\alpha))$ we can write

$$\left|\underline{\psi}_i(t_m, w^{[m]}(\alpha), 0) - \underline{f}_i(t_m + \underline{\theta}_i h, y(t_m + \underline{\theta}_i h\right|$$

$$\leq hL_1\underline{\theta}_i + L_1\sum_{i=1}^{n}\left|\underline{y}_i(t_m + \underline{\theta}_i h) - \underline{y}_i(t_m)\right| + L_1\sum_{i=1}^{n}\left|\overline{y}_i(t_m + \overline{\theta}_i h) - \overline{y}_i(t_m)\right|$$

$$= hL_1\underline{\theta}_i + L_1\sum_{i=1}^{n}\left|\underline{y}_i'(t_m + \xi_i\underline{\theta}_i h) - \underline{\theta}_i h\right| + L_1\sum_{i=1}^{n}\left|\overline{y}_i'(t_m + \xi_i\overline{\theta}_i h)\underline{\theta}_i h\right| = hL_2,$$

then

$$\left|\underline{e}^{[m+1]}(\alpha)\right| \leq \left|\underline{e}_i^{[m]}(\alpha)\right| + hL_1\left\{\sum_{j=1}^{n}\left|\underline{e}_j^{[m]}(\alpha)\right| + \left|\overline{e}_j^{[m]}(\alpha)\right|\right\} + h^2 L_1 + h^2 L_2$$

$$\leq \left|\underline{e}_j^{[m]}(\alpha)\right| + nhL_1 \max_{1\leq j\leq n}\left\{\left|\overline{e}_j^{[m]}(\alpha)\right| + h^2(L_1 + L_2)\right.$$

On the other hand

$$\max_{1\leq j\leq n}\left\{\left|\underline{e}_j^{[m]}(\alpha)\right|\right\} = k_i\left|\underline{e}_i^{[m]}(\alpha)\right|, \quad \max_{1\leq j\leq n}\left\{\left|\overline{e}_j^{[m]}(\alpha)\right|\right\} = k_i'\left|\overline{e}_i^{[m]}(\alpha)\right|,$$

with assumption $k_1 = \max_{1\leq i\leq n}\left\{k_i, k_i'\right\}$ and $M = L_1 + L_2$, we can write

$$\begin{aligned}\left|\underline{e}_i^{[m+1]}(\alpha)\right| &\leq \left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\underline{e}_i^{[m]}(\alpha)\right| + nhk_1L_1\left\{\left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\overline{e}_i^{[m]}(\alpha)\right|\right\} + M_1h^2 \\ &\leq \left|\underline{e}_i^{[m]}(\alpha)\right| + 2nhk_1L_1\left\{\left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\overline{e}_i^{[m]}(\alpha)\right|\right\} + M_1h^2,\end{aligned} \tag{5.36}$$

similarly, we can obtain the following relation

$$\left|\underline{e}_i^{[m+1]}(\alpha)\right| \leq \left|\underline{e}_i^{[m]}(\alpha)\right| 2nhk_2L_1' \max\left\{\left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\overline{e}_i^{[m]}(\alpha)\right|\right\} + M_2h^2 \tag{5.37}$$

Now, we input $L = \max\{L_1, L_1'\}$ and $M = \max\{M_1, M_2\}$ the relations (5.36) and (5.37) can be written as follow

$$\left|\underline{e}_1^{[m+1]}(\alpha)\right| \leq \left|\underline{e}_i^{[m]}(\alpha)\right| + 2nhkL \max\left\{\left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\overline{e}_i^{[m]}(\alpha)\right|\right\} + Mh^2,$$

$$\left|\overline{e}_i^{[m+1]}(\alpha)\right| \leq \left|\overline{e}_i^{[m]}(\alpha)\right| + 2nhkL \max\left\{\left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\overline{e}_i^{[m]}(\alpha)\right|\right\} + Mh^2$$

Denote $e_i^{[m]} = \left|\underline{e}_i^{[m]}(\alpha)\right| + \left|\overline{e}_i^{[m]}(\alpha)\right|$. Then By virtue of lemma (5.7)

$$e_i^{[m]}(\alpha) \leq (1 + 4nhkL)^m e_i^{[0]}(\alpha) + 2Mh^2\frac{(1+4nhkL)^m - 1}{4nhkL},$$

where

$$e_i^{[0]} = \left|\underline{e}_i^{[0]}(\alpha)\right| + \left|\overline{e}_i^{[0]}(\alpha)\right|. \text{Then}$$

$$\left|\underline{e}_i^{[m]}(\alpha)\right| \leq \underline{e}^{4mnkh} \times e_i^{[0]} + M\frac{e^{4mnklh}}{2nhkl}h \text{ and } \left|\overline{e}_i^{[m]}(\alpha)\right| \leq e^{4mnkh} \times e_i^{[0]} + M\frac{e^{4mnklh}}{2nhkL}h.$$

In particular

$$\left|\underline{e}_i^{[N]}(\alpha)\right| \leq \underline{e}^{4Nnkh} \times e_i^{[0]} + M\frac{e^{4Nnklh}}{2nhkl}h \text{ and } \left|\overline{e}_i^{[N]}(\alpha)\right| \leq e^{4Nnkh} \times e_i^{[0]} + M\frac{e^{4Nnklh}}{2nhkL}h.$$

Since

$\underline{e}_i^{[0]}(\alpha)=\overline{e}_i^{[0]}(\alpha)=0,$ and $h=\dfrac{T}{N}$ we obtain

Then $\left\|e_1^{[N]}(\alpha)\right\|\le M\dfrac{e^{4Nnklh}}{2nhkL}h,$ and $\left\|e_2^{[N]}(\alpha)\right\|\le M\dfrac{e^{4Nnklh}}{2nhkL}h.$ Then $\left\|e^{[N]}(\alpha)\right\|\le 2M\dfrac{e^{4Nnklh}}{2nhkL}h,$ if $h\to 0$ we get $\left\|e^{[N]}(\alpha)\right\|\to 0,$ so the numerical solution (5.29) converge to the solutions (5.31). Conversely, suppose that the numerical method (5.29) convergent to the solution of the system (5.31). With absurd hypothesis we suppose that (5.30) is not correct. Then $\psi\left(t,y(t,\alpha),0\right)=g\left(t,y(t,\alpha)\right)\neq F\left(t,y(t,\alpha)\right).$ Similarly, we can prove that the numerical method of (5.29) is convergent to the solution of following system

$$\begin{cases} u'(t))=g(t,y(t)),\\ u(0)=y^{[0]}(\alpha),\end{cases}$$

Then

$y(t,\alpha)=u(t).$ Since $g(t,y(t,\alpha))\neq F(t,y(t,\alpha)),$ suppose that $\boldsymbol{F}$ and $\boldsymbol{g}$ differ at some point $(t_a,y(t_a,\alpha)).$ If we consider the initial values problem (5.31) and (5.38) starting from $(t_a,y(t_a,\alpha))$ we have $y'(t_a,\alpha)=F(t_a,y(t_a,\alpha))\neq g(t_a,y(t_a,\alpha))=g(t_a,u(t_a))=u'(t_a),$ which is a contradiction.

Corollary 5.1

The proposed Runge-Kutta method based on Contra-harmonic Mean by (4.24) and is convergent to the solution of the system (3.13) respectively.

6. NUMERICAL EXAMPLE

Example 6.1

Consider the following fuzzy differential equation with fuzzy initial value

$$\begin{cases} y''(t)-4y'(t)+4y(t)=0 \qquad (t\ge 0)\\ y(0)=(2+\alpha,4-\alpha)\\ y'(0)=(5+\alpha,7-\alpha)\end{cases}$$

The exact solution is as follows:

$$\underline{y}(t,r)=(2+r)e^{2t}+(1-r)te^{2t})$$
$$\overline{y}(t,r)=(4-r)e^{2t}+(r-1)te^{2t})$$

The solution of the fourth order Runge - Kutta method based on Contra-harmonic Mean is as follows:

$$\underline{w}_1^{[m+1]}(\alpha)=\underline{w}_1^{[m]}(\alpha)+$$
$$\frac{\underline{w}_2{}^2[\frac{2}{3}h+\frac{8}{3}h^2+\frac{25}{6}h^3+3h^4+\frac{27}{16}h^5]+\underline{w}_1{}^2[\frac{8}{3}h^3+4h^4+3h^5]-\underline{w}_1\underline{w}_2[\frac{8}{3}h^2+\frac{22}{3}h^3+7h^4+\frac{9}{2}h^5]}{\underline{w}_2[2+4h+\frac{9}{4}h^2]-\underline{w}_1[4h+3h^2]}+$$
$$\frac{\underline{w}_2{}^2[\frac{2}{3}h+\frac{4}{3}h^2+\frac{4}{3}h^3]+\underline{w}_1{}^2[\frac{4}{3}h^3]-\underline{w}_1\underline{w}_2[\frac{4}{3}h^2+\frac{8}{3}h^3]}{\underline{w}_2[2+2h]-2h\underline{w}_1}+$$
$$\frac{\underline{w}_2{}^2[\frac{2}{3}h+4h^2+\frac{67}{6}h^3+21h^4+\frac{519}{16}h^5+27h^6+27h^7]+\underline{w}_1{}^2[\frac{20}{3}h^3+20h^4+51h^5+54h^6+\frac{243}{4}h^7]-\underline{w}_1\underline{w}_2[4h^2+\frac{58}{3}h^3+44h^4+\frac{165}{2}h^5+\frac{153}{2}h^6+81h^7]}{\underline{w}_2[2+6h+\frac{27}{4}h^2+9h^3]-\underline{w}_1[6h+9h^2+\frac{27}{2}h^3]}$$

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$$\overline{w}_1^{[m+1]}(\alpha)=\overline{w}_1^{[m]}(\alpha)+$$

$$\frac{\overline{w}_2^{\,2}[\frac{2}{3}h+\frac{8}{3}h^2+\frac{25}{6}h^3+3h^4+\frac{27}{16}h^5]+\overline{w}_1^{\,2}[\frac{8}{3}h^3+4h^4+3h^5]-\overline{w}_1\overline{w}_2[\frac{8}{3}h^2+\frac{22}{3}h^3+7h^4+\frac{9}{2}h^5]}{\overline{w}_2[2+4h+\frac{9}{4}h^2]-\overline{w}_1[4h+3h^2]}+$$

$$\frac{\overline{w}_2^{\,2}[\frac{2}{3}h+\frac{4}{3}h^2+\frac{4}{3}h^3]+\overline{w}_1^{\,2}[\frac{4}{3}h^3]-\overline{w}_1\overline{w}_2[\frac{4}{3}h^2+\frac{8}{3}h^3]}{\overline{w}_2[2+2h]-2h\overline{w}_1}+$$

$$\frac{\overline{w}_2^{\,2}[\frac{2}{3}h+4h^2+\frac{67}{6}h^3+21h^4+\frac{519}{16}h^5+27h^6+27h^7]+\overline{w}_1^{\,2}[\frac{20}{3}h^3+20h^4+51h^5+54h^6+\frac{243}{4}h^7]-\overline{w}_1\overline{w}_2[4h^2+\frac{58}{3}h^3+44h^4+\frac{165}{2}h^5+\frac{153}{2}h^6+81h^7]}{\overline{w}_2[2+6h+\frac{27}{4}h^2+9h^3]-\overline{w}_1[6h+9h^2+\frac{27}{2}h^3]}$$

$$\underline{w}_2^{[m+1]}(\alpha)=\underline{w}_2^{[m]}(\alpha)+$$

$$\frac{\underline{w}_2^{\,2}[\frac{32}{3}h+16h^2+12h^3]+\underline{w}_1^{\,2}[\frac{32}{3}h+\frac{64}{3}h^2+\frac{64}{3}h^3]-\underline{w}_2\underline{w}_1[\frac{64}{3}h+\frac{112}{3}h^2+32h^3]}{\underline{w}_2[8+6h]-\underline{w}_1[8+8h]}+$$

$$\frac{\underline{w}_2^{\,2}[\frac{32}{3}h+32h^2+40h^3+24h^4+12h^5]+\underline{w}_1^{\,2}[\frac{32}{3}h+\frac{128}{3}h^2+\frac{200}{3}h^3+48h^4+27h^5]-\underline{w}_2\underline{w}_1[\frac{64}{3}h+\frac{224}{3}h^2+104h^3+68h^4+36h^5]}{\underline{w}_2[8+12h+6h^2]-\underline{w}_1[8+16h+9h^2]}+$$

$$\frac{\underline{w}_2^{\,2}[\frac{32}{3}h+48h^2+108h^3+180h^4+240h^5+180h^6+\frac{675}{4}h^7]+\underline{w}_1^{\,2}[48+\frac{486}{3}h^3+\frac{19303}{9}h^4+\frac{85264}{81}h^6]-\underline{w}_2\underline{w}_1[\frac{64}{3}h+112h^2+280h^3+496h^4+708h^5+558h^6+540h^7]}{\underline{w}_2[8+18h+18h^2+\frac{45}{2}h^3]-\underline{w}_1[8+24h+27h^2+36h^3]}+$$

$$\overline{w}_2^{[m+1]}(\alpha)=\overline{w}_2^{[m]}(\alpha)+$$

$$\frac{\overline{w}_2^{\,2}[\frac{32}{3}h+16h^2+12h^3]+\overline{w}_1^{\,2}[\frac{32}{3}h+\frac{64}{3}h^2+\frac{64}{3}h^3]-\overline{w}_2\overline{w}_1[\frac{64}{3}h+\frac{112}{3}h^2+32h^3]}{\overline{w}_2[8+6h]-\overline{w}_1[8+8h]}+$$

$$\frac{\overline{w}_2^{\,2}[\frac{32}{3}h+32h^2+40h^3+24h^4+12h^5]+\overline{w}_1^{\,2}[\frac{32}{3}h+\frac{128}{3}h^2+\frac{200}{3}h^3+48h^4+27h^5]-\overline{w}_2\overline{w}_1[\frac{64}{3}h+\frac{224}{3}h^2+104h^3+68h^4+36h^5]}{\overline{w}_2[8+12h+6h^2]-\overline{w}_1[8+16h+9h^2]}+$$

$$\frac{\overline{w}_2^{\,2}[\frac{32}{3}h+48h^2+108h^3+180h^4+240h^5+180h^6+\frac{675}{4}h^7]+\overline{w}_1^{\,2}[48+\frac{486}{3}h^3+\frac{19303}{9}h^4+\frac{85264}{81}h^6]-\overline{w}_2\overline{w}_1[\frac{64}{3}h+112h^2+280h^3+496h^4+708h^5+558h^6+540h^7]}{\overline{w}_2[8+18h+18h^2+\frac{45}{2}h^3]-\overline{w}_1[8+24h+27h^2+36h^3]}+$$

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Table 1 shows the obtained results by the proposed method for r=0 to 1 and compared with the fourth order Runge – Kutta method(Table 2)

Table 1 (for ex 6.1)

FIVPRKCoM4SYS2							
h=0.1, t=1		h=0.1, t=2		h=0.01, t=1		h=0.01, t=2	
error in $\underline{w}$	error in $\overline{w}$	error in $\underline{w}$	error in $\overline{w}$	error in $\underline{w}$	error in $\overline{w}$	error in $\underline{w}$	error in $\overline{w}$
2.79E-03	1.99E-03	3.98E-02	3.85E-02	1.70E-05	2.36E-05	1.82E-04	5.23E-04
2.49E-03	1.68E-03	3.63E-02	3.02E-02	1.41E-05	1.88E-05	1.56E-04	3.94E-04
2.21E-03	1.43E-03	3.29E-02	2.37E-02	1.13E-05	1.46E-05	1.30E-04	2.92E-04
1.95E-03	1.22E-03	2.96E-02	1.88E-02	8.85E-06	1.10E-05	1.05E-04	2.10E-04
1.71E-03	1.05E-03	2.64E-02	1.52E-02	6.64E-06	7.96E-06	8.15E-05	1.46E-04
1.50E-03	9.19E-04	2.34E-02	1.27E-02	4.73E-06	5.46E-06	6.00E-05	9.59E-05
1.31E-03	8.29E-04	2.06E-02	1.12E-02	3.12E-06	3.47E-06	4.10E-05	5.86E-05
1.15E-03	7.76E-04	1.80E-02	1.05E-02	1.84E-06	1.96E-06	2.50E-05	3.18E-05
1.01E-03	7.58E-04	1.57E-02	1.05E-02	8.99E-07	9.14E-07	1.27E-05	1.42E-05
9.01E-04	7.74E-04	1.37E-02	1.10E-02	3.17E-07	3.05E-07	4.64E-06	4.55E-06
8.22E-04	8.22E-04	1.21E-02	1.21E-02	1.12E-07	1.12E-07	1.66E-06	1.66E-06

Table 2 (for ex 6.1)

FIVPRK4SYS2							
h=0.1, t=1		h=0.1, t=2		h=0.01, t=1		h=0.01, t=2	
error in $\underline{w}$	error in $\overline{w}$	error in $\underline{w}$	error in $\overline{w}$	error in $\underline{w}$	error in $\overline{w}$	error in $\underline{w}$	error in $\overline{w}$
1.25E-05	4.09E-06	1.58E-02	1.03E-03	1.06E-07	9.85E-09	1.86E-06	1.41E-07
1.20E-05	4.51E-06	1.50E-02	1.86E-04	1.02E-07	1.47E-08	1.76E-06	4.08E-08
1.16E-05	4.93E-06	1.41E-02	6.57E-04	9.68E-08	1.95E-08	1.66E-06	5.92E-08
1.12E-05	5.35E-06	1.33E-02	1.50E-03	9.19E-08	2.43E-08	1.56E-06	1.59E-07
1.08E-05	5.76E-06	1.25E-02	2.34E-03	8.71E-08	2.92E-08	1.46E-06	2.59E-07
1.04E-05	6.18E-06	1.16E-02	3.18E-03	8.23E-08	3.40E-08	1.36E-06	3.59E-07
9.95E-06	6.60E-06	1.08E-02	4.03E-03	7.74E-08	3.88E-08	1.26E-06	4.59E-07
9.53E-06	7.02E-06	9.93E-03	4.87E-03	7.26E-08	4.37E-08	1.16E-06	5.59E-07
9.11E-06	7.44E-06	9.08E-03	5.71E-03	6.78E-08	4.85E-08	1.06E-06	6.59E-07
8.69E-06	7.86E-06	8.24E-03	6.55E-03	6.30E-08	5.33E-08	9.59E-07	7.59E-07
8.27E-06	8.27E-06	7.40E-03	7.40E-03	5.81E-08	5.81E-08	8.59E-07	8.59E-07

Example 6.2

Consider the following fuzzy differential equation with fuzzy initial value

$$\begin{cases} y'''(t) = 2y''(t) + 3y'(t) \qquad (0 \le t \le 1) \\ y(0) = (3+\alpha, 5-\alpha) \\ y'(0) = (-3+\alpha, -1-\alpha) \\ y''(0) = (8+\alpha, 10-\alpha) \end{cases}$$

the eigenvalue-eigenvector solution is as follows:

$$y(t,r) = \left(-\frac{1}{3} + \frac{7}{12}e^{3t} + \left(\frac{11}{4} + r\right)e^{-t}, -\frac{1}{3} + \frac{7}{12}e^{3t} + \left(\frac{19}{4} - r\right)e^{-t}\right).$$

The solution of the fourth order Runge - Kutta method based on Contra-harmonic Mean is as follows :

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$$\underline{w}_1^{[m+1]}(\alpha)=\underline{w}_1^{[m]}(\alpha)+\frac{\underline{w}_2^{\,2}[\frac{2}{3}h+\frac{3}{8}h^3+\frac{27}{256}h^5]+\underline{w}_3^{\,2}[\frac{1}{3}h^3+\frac{1}{8}h^4+\frac{3}{64}h^5]+\underline{w}_2\underline{w}_3[\frac{2}{3}h^2+\frac{1}{4}h^3+\frac{3}{16}h^4+\frac{9}{64}h^5]}{\underline{w}_2[2+\frac{9}{16}h^2]+\underline{w}_3[h+\frac{3}{8}h^2]}+$$

$$\frac{\underline{w}_2^{\,2}[\frac{2}{3}h+\frac{9}{8}h^3+\frac{9}{8}h^4+\frac{135}{256}h^5+\frac{81}{64}h^6+\frac{243}{256}h^7]+\underline{w}_3^{\,2}[\frac{5}{12}h^3+\frac{5}{8}h^4+\frac{99}{64}h^5+\frac{63}{64}h^6+\frac{1323}{1024}h^7]+\underline{w}_2\underline{w}_3[h^2+\frac{3}{4}h^3+\frac{9}{4}h^4+\frac{117}{64}h^5+\frac{297}{128}h^6+\frac{567}{256}h^7]}{\underline{w}_2[2+\frac{27}{16}h^2+\frac{27}{16}h^3]+\underline{w}_3[\frac{3}{2}h+\frac{9}{8}h^2+\frac{63}{32}h^3]}+$$

$$\frac{\frac{2h}{3}\underline{w}_2^{\,2}+\frac{h^3}{12}\underline{w}_3^{\,2}+\frac{h^2}{3}\underline{w}_2\underline{w}_3}{2\underline{w}_2+\frac{h}{2}\underline{w}_3}$$

$$\overline{w}_1^{[m+1]}(\alpha)=\overline{w}_1^{[m]}(\alpha)+\frac{\overline{w}_2^{\,2}[\frac{2}{3}h+\frac{3}{8}h^3+\frac{27}{256}h^5]+\overline{w}_3^{\,2}[\frac{1}{3}h^3+\frac{1}{8}h^4+\frac{3}{64}h^5]+\overline{w}_2\overline{w}_3[\frac{2}{3}h^2+\frac{1}{4}h^3+\frac{3}{16}h^4+\frac{9}{64}h^5]}{\overline{w}_2[2+\frac{9}{16}h^2]+\overline{w}_3[h+\frac{3}{8}h^2]}+$$

$$\frac{\overline{w}_2^{\,2}[\frac{2}{3}h+\frac{9}{8}h^3+\frac{9}{8}h^4+\frac{135}{256}h^5+\frac{81}{64}h^6+\frac{243}{256}h^7]+\overline{w}_3^{\,2}[\frac{5}{12}h^3+\frac{5}{8}h^4+\frac{99}{64}h^5+\frac{63}{64}h^6+\frac{1323}{1024}h^7]+\overline{w}_2\overline{w}_3[h^2+\frac{3}{4}h^3+\frac{9}{4}h^4+\frac{117}{64}h^5+\frac{297}{128}h^6+\frac{567}{256}h^7]}{\overline{w}_2[2+\frac{27}{16}h^2+\frac{27}{16}h^3]+\overline{w}_3[\frac{3}{2}h+\frac{9}{8}h^2+\frac{63}{32}h^3]}+$$

$$\frac{\frac{2h}{3}\overline{w}_2^{\,2}+\frac{h^3}{12}\overline{w}_3^{\,2}+\frac{h^2}{3}\overline{w}_2\overline{w}_3}{2\overline{w}_2+\frac{h}{2}\overline{w}_3}$$

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$$\underline{w}_2^{[m+1]}(\alpha) = \underline{w}_2^{[m]}(\alpha) + \frac{\underline{w}_2^{2}[\frac{15}{4}h^3 + \frac{45}{8}h^4 + \frac{189}{64}h^5 + \frac{567}{64}h^6 + \frac{11907}{1024}h^7] + \underline{w}_3^{2}[\frac{2}{3}h + 2h^2 + \frac{343}{72}h^3 + \frac{226}{27}h^4 + \frac{33301}{3456}h^5 + \frac{140525}{15552}h^6 + \frac{133225}{23328}h^7] + \underline{w}_2\underline{w}_3[3h^2 + \frac{29}{4}h^3 + \frac{57}{4}h^4 + \frac{1539}{64}h^5 + \frac{2403}{128}h^6 + \frac{2835}{128}h^7]}{\underline{w}_2[\frac{9}{2}h + \frac{27}{8}h^2 + \frac{189}{32}h^3] + \underline{w}_3[2 + 3h + \frac{63}{16}h^2 + \frac{45}{8}h^3]} + \frac{\underline{w}_2^{2}[\frac{3}{4}h^3] + \underline{w}_3^{2}[\frac{2}{3}h + \frac{2}{3}h^2 + \frac{1}{3}h^3] + \underline{w}_2\underline{w}_3[h^2 + h^3]}{\frac{3}{2}h\underline{w}_2 + [2+h]\underline{w}_3} + \frac{\underline{w}_2^{2}[\frac{3}{2}h^3 + \frac{9}{8}h^4 + \frac{27}{64}h^5] + \underline{w}_3^{2}[h^2 + \frac{11}{4}h^3 + \frac{33}{16}h^4 + \frac{63}{64}h^5] + \underline{w}_2\underline{w}_3[h^2 + \frac{11}{4}h^3 + \frac{33}{16}h^4 + \frac{63}{64}h^5]}{\underline{w}_2[3h + \frac{9}{8}h^2] + \underline{w}_3[2 + 2h + \frac{21}{16}h^2]} +$$

$$\overline{w}_2^{[m+1]}(\alpha) = \overline{w}_2^{[m]}(\alpha) + \frac{\overline{w}_2^{2}[\frac{15}{4}h^3 + \frac{45}{8}h^4 + \frac{189}{64}h^5 + \frac{567}{64}h^6 + \frac{11907}{1024}h^7] + \overline{w}_3^{2}[\frac{2}{3}h + 2h^2 + \frac{343}{72}h^3 + \frac{226}{27}h^4 + \frac{33301}{3456}h^5 + \frac{140525}{15552}h^6 + \frac{133225}{23328}h^7] + \overline{w}_2\overline{w}_3[3h^2 + \frac{29}{4}h^3 + \frac{57}{4}h^4 + \frac{1539}{64}h^5 + \frac{2403}{128}h^6 + \frac{2835}{128}h^7]}{\overline{w}_2[\frac{9}{2}h + \frac{27}{8}h^2 + \frac{189}{32}h^3] + \overline{w}_3[2 + 3h + \frac{63}{16}h^2 + \frac{45}{8}h^3]} + \frac{\overline{w}_2^{2}[\frac{3}{4}h^3] + \overline{w}_3^{2}[\frac{2}{3}h + \frac{2}{3}h^2 + \frac{1}{3}h^3] + \overline{w}_2\overline{w}_3[h^2 + h^3]}{\frac{3}{2}h\overline{w}_2 + [2+h]\overline{w}_3} + \frac{\overline{w}_2^{2}[\frac{3}{2}h^3 + \frac{9}{8}h^4 + \frac{27}{64}h^5] + \overline{w}_3^{2}[h^2 + \frac{11}{4}h^3 + \frac{33}{16}h^4 + \frac{63}{64}h^5] + \overline{w}_2\overline{w}_3[h^2 + \frac{11}{4}h^3 + \frac{33}{16}h^4 + \frac{63}{64}h^5]}{\overline{w}_2[3h + \frac{9}{8}h^2] + \overline{w}_3[2 + 2h + \frac{21}{16}h^2]}$$

$$\underline{w}_3^{[m+1]}(\alpha) = \underline{w}_3^{[m]}(\alpha) +$$
$$\underline{w}_2^{\,2}[6h+18h^2+\frac{309}{8}h^3+\frac{585}{8}h^4+\frac{23895}{256}h^5+\frac{2835}{32}h^6+\frac{6075}{64}h^7]+$$
$$\underline{w}_3^{\,2}[\frac{8}{3}h+14h^2+\frac{425}{12}h^3+\frac{533}{8}h^4+\frac{207}{2}h^5+\frac{2745}{32}h^6+\frac{100467}{1024}h^7]+$$
$$\frac{\underline{w}_2\underline{w}_3[8h+33h^2+\frac{293}{4}h^3+\frac{561}{4}h^4+\frac{6291}{32}h^5+\frac{22329}{128}h^6+\frac{24705}{128}h^7]}{\underline{w}_2[9h+\frac{189}{16}h^2+\frac{135}{8}h^3]+\underline{w}_3[4+\frac{21}{2}h+\frac{45}{4}h^2+\frac{549}{32}h^3]}+$$
$$\frac{\underline{w}_2^{\,2}[12h+12h^2+\frac{111}{8}h^3+\frac{63}{8}h^4+\frac{1323}{256}h^5]+\underline{w}_3^{\,2}[\frac{8}{3}h+\frac{28}{3}h^2+\frac{79}{6}h^3+\frac{35}{4}h^4+\frac{75}{16}h^5]+\underline{w}_2\underline{w}_3[8h+22h^2+\frac{107}{4}h^3+\frac{267}{16}h^4+\frac{315}{32}h^5]}{\underline{w}_2[6+6h+\frac{63}{16}h^2]+\underline{w}_3[4+7h+\frac{15}{4}h^2]}+$$
$$\frac{\underline{w}_2^{\,2}[6h^2+3h^3]+\underline{w}_3^{\,2}[\frac{8}{3}h+\frac{14}{3}h^2+\frac{49}{12}h^3]+\underline{w}_2\underline{w}_3[8h+11h^2+7h^3]}{\underline{w}_2[6+3h]+\underline{w}_3[4+\frac{7}{2}h]}+$$

$$\overline{w}_3^{[m+1]}(\alpha) = \overline{w}_3^{[m]}(\alpha) +$$
$$\overline{w}_2^{\,2}[6h+18h^2+\frac{309}{8}h^3+\frac{585}{8}h^4+\frac{23895}{256}h^5+\frac{2835}{32}h^6+\frac{6075}{64}h^7]+$$
$$\overline{w}_3^{\,2}[\frac{8}{3}h+14h^2+\frac{425}{12}h^3+\frac{533}{8}h^4+\frac{207}{2}h^5+\frac{2745}{32}h^6+\frac{100467}{1024}h^7]+$$
$$\frac{\overline{w}_2\overline{w}_3[8h+33h^2+\frac{293}{4}h^3+\frac{561}{4}h^4+\frac{6291}{32}h^5+\frac{22329}{128}h^6+\frac{24705}{128}h^7]}{\overline{w}_2[9h+\frac{189}{16}h^2+\frac{135}{8}h^3]+\overline{w}_3[4+\frac{21}{2}h+\frac{45}{4}h^2+\frac{549}{32}h^3]}+$$
$$\frac{\overline{w}_2^{\,2}[12h+12h^2+\frac{111}{8}h^3+\frac{63}{8}h^4+\frac{1323}{256}h^5]+\overline{w}_3^{\,2}[\frac{8}{3}h+\frac{28}{3}h^2+\frac{79}{6}h^3+\frac{35}{4}h^4+\frac{75}{16}h^5]+\overline{w}_2\overline{w}_3[8h+22h^2+\frac{107}{4}h^3+\frac{267}{16}h^4+\frac{315}{32}h^5]}{\overline{w}_2[6+6h+\frac{63}{16}h^2]+\overline{w}_3[4+7h+\frac{15}{4}h^2]}+$$
$$\frac{\overline{w}_2^{\,2}[6h^2+3h^3]+\overline{w}_3^{\,2}[\frac{8}{3}h+\frac{14}{3}h^2+\frac{49}{12}h^3]+\overline{w}_2\overline{w}_3[8h+11h^2+7h^3]}{\overline{w}_2[6+3h]+\overline{w}_3[4+\frac{7}{2}h]}+$$

Figure 1 shows the obtained results and compared with the fourth order Runge – Kutta method for h=0.01 and t = 1.

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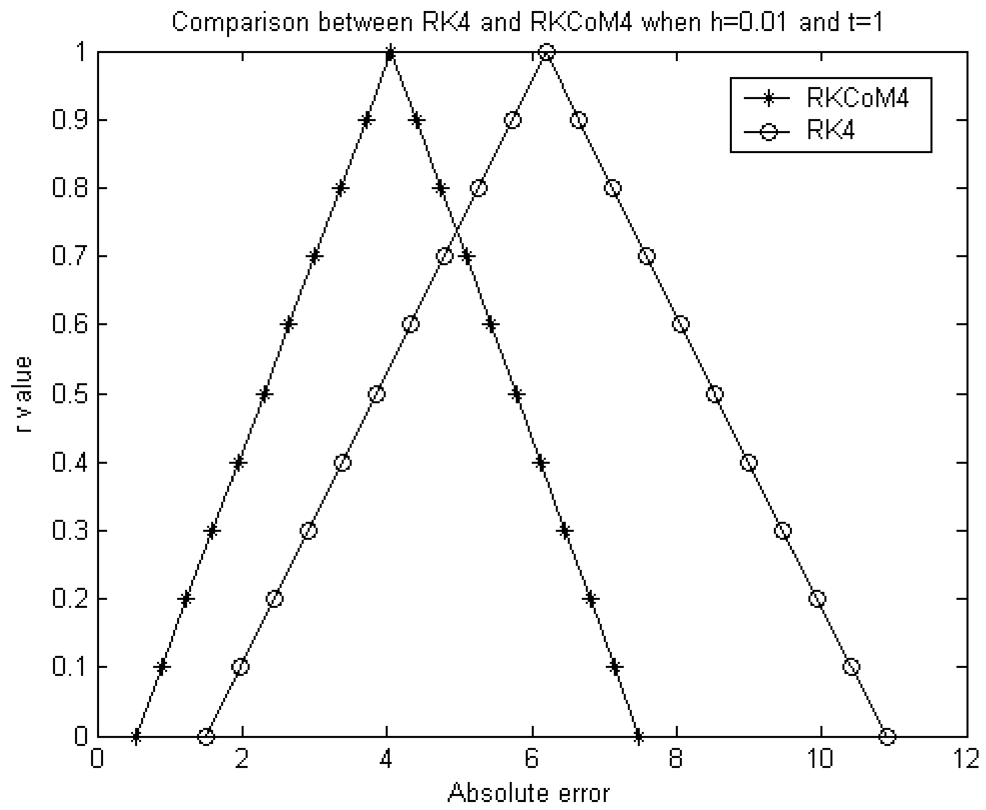

Figure 1 (for ex.6.2 h=0.01 and t=1)

7. CONCLUSION

In this paper a numerical method for solving N^{th} - order fuzzy initial value problem is presented. In this method N^{th} - order fuzzy linear differential equation is converted to a fuzzy system which will be solved with the fourth order Runge-Kutta method based on Contra-harmonic Mean. From the numerical examples 6.1 and 6.2, by comparing the absolute error values at t=1 and t=2 i.e. from the tables 1 and 2 it is concluded that the proposed method gives a good accuracy for solving the FIVPs. The proposed method also compared with the classical fourth order Runge – Kutta method and presented in table 2 and figure 2.

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