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A Quadruple Coincidence Point Theorem for Two Mappings for Implicit Relation Satisfying O-Compatible Condition

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***Abstract*:** The result in the present paper is a quadruple coincidence point theorem in partially ordered complete metric space for implicit relation in two mappings satisfying o-compatible condition. As o-compatible condition is a more generalised condition than compatibility condition thus the result of this paper extends and generalises many results of coupled and tripled coincidence point theorem available in the literature.

Keywords: *partially ordered complete metric space, implicit relation, o-compatible condition, mixed monotone property.*

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1. INTRODUCTION:

Fixed point theory is one of the fundamental and very efficient tools in nonlinear functional analysis as its ever-growing use in this field is very widely spread with its applications. In particular, the results of fixed point theory are most apparent in fields like economics, computer sciences and engineering including many branches of mathematics.

The most considerable advances in fixed point theory started after the distinguished fixed point result of Banach, known as Banach's contraction mapping principle.

Further in the light of these developments, the concept of coupled fixed point was introduced by Guo and Lakshmikantham[4] in 1987.Later, Bhaskar and Lakshmikantham [17] gave the idea of mixed monotone mapping and proved some coupled fixed point theorems for the mixed monotone mappings.Lakshmikantham et.al. [19] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings in partially ordered complete metric spaces. Soon after, many results [2, 5, 6, 11, 12, 14] on coupled fixed point have been obtained.

V.Berinde et.al. [18] proposed the idea of a tripled fixed point.

B. Samet et.al. [3] put forward for the first time fixed point of order $N \geq 3$.Many researchers [7-10] were motivated and proved theorems on quadruple fixed points with monotone property.

A step ahead in the line of generalisation Popa [20] introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. In nonlinear analysis, especially in fixed point theory, research articles[13,16] on implicit relations on metric spaces have been studied.

To prove the existence and uniqueness of coupled coincidence point or coupled fixed for two or more mappings the relation between the two maps is required. Choudhury and Kundu[1] defined the notion of compatibility point . Later, Luong and Thuan [15] slightly improved the notion of compatible mappings on partially ordered metric spaces, namely O-compatible mappings. It can be proved that in a partially metric space if two mappings are compatible then they are O-compatible. However, the converse is not true.

The result in the present paper is an extension and generalisations of many results of coupled coincidence point or coupled fixed theorem to quadruple coincidence point theorem for implicit relation in two mappings satisfying o-compatible condition in partially ordered complete metric space.

2. PRELIMINARY

We need to use the following fundamental concepts throughout this paper.

Definitions:

2.1: Convergent Sequence: Let (X, d) be a metric space. The sequence $\{x_n\}$ in X is said to be a convergent sequence if for every $0 < \in$, there is $n_0 \in N$ such that for all $n \geq n_0$, $d(x_n, x) < \in$ for some $x \in X$. We denote this by $\lim_{n\to\infty} x_n = x$.

2.2: Cauchy Sequence: Let (X, d) be a metric space. The sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for all $0 \ll \in$, there is $n_o \in N$ such that $d(x_m, x_n) \ll \in$, for all $m, n \geq n_0$.

2.3: Complete metric space: A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

2.4:-O-compatible mapping: Let $(X, d, \leq)$ be a partially ordered metric space. The mappings $A: X \times X \to X$ and $f: X \to X$ are said to be O-compatible if $\lim_{n\to\infty} d\left(fA(x_n, y_n), A(fx_n, fy_n)\right) = 0$ and $\lim_{n\to\infty} d\left(fA(y_n, x_n), A(fy_n, fx_n)\right) = 0$ where $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\{fx_n\}$ $\{fy_n\}$ are monotone and $\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} fy_n = y$ for all $x, y \in X$ are satisfied.

2.5: Let R^+ denote the set of all nonnegative real numbers. Also, let Φ denote the collection of all functions $\xi : R^+ \to R^+$ which satisfy

(i) ξ is continuous and non-decreasing,

(ii) $\xi(t) < t$ for $t > 0$ and $\xi(0) = 0$

2.6: Υ for the class of all continuous functions $H : R_+^{10} \to R$ satisfying

(H1) $H(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10})$ is non increasing in t_5 and t_{10}.

(H2) $H(a, b, c, d, d+e, e, f, g, h, d+e) \leq 0$ then $(a+b+c+d) \leq \lambda\,(e+f+g+h)$ where $\lambda \in [0,1)$.

2.7: Example: The following functions lie in H:

1. $H(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = t_4 - \alpha t_5 - \beta t_6 - \delta t_7 - \theta t_8 - \omega t_9 - \lambda t_{10}$ where $\alpha, \beta, \delta, \theta, \omega, \lambda > 0$ and $\alpha + \beta + \delta + \theta + \omega + \lambda < 1$.
2. $H(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = t_4 - \frac{\alpha}{4}\left(t_6 + t_7 + t_8 + t_9\right)$ where $\alpha \in (0,1)$
3. $H(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = t_1 + t_2 + t_3 + t_4 - \alpha(t_6 + t_7 + t_8 + t_9)$ where $\alpha \in (0,1)$
4. $H(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = t_4 - \xi\left(\frac{t_6 + t_7 + t_8 + t_9}{4}\right)$

3. MAIN THEOREM

The following result is a quadruple coincidence point theorem for two mappings for implicit relation satisfying o-compatible condition.

3.1-Theorem: Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose that $A: X\times X\times X\times X \to X$ and $f: X \to X$ are two mappings such that A has the mixed g-monotone property. Assume that there exists $H \in \Upsilon$ such that

$$\text{(i)}\ H\begin{pmatrix} (d(A(w,x,y,z),A(s,p,q,r)),(d(A(z,w,x,y),A(r,s,p,q)),(d(A(y,z,w,x),A(q,r,s,p)),\\ (d(A(x,y,z,w),A(p,q,r,s)),((d(A(x,y,z,w),f(x))+(d(A(p,q,r,s),f(p))),\\ (d(f(x),f(p))),(d(f(y),f(q))),(d(f(z),f(r))),(d(f(w),f(s))),\\ ((d(A(x,y,z,w),f(p))+(d(A(p,q,r,s),f(x))) \end{pmatrix} \leq 0$$

for all x, y, z, w, p, q, r, s $\in$ X with $f(x)\geq f(p),\ f(y)\leq f(q),\ f(z)\geq f(r),\ f(w)\leq f(s)$. Also $A(X\times X\times X\times X)\subseteq f(x)$ and g is continuous on X and O-compatible with A. Additionally, suppose that either
(a) A is continuous, or
(b) X has the properties
(i) If a non-decreasing sequence $fx_n \to a$, then $fx_n \leq a$ for all n,
(ii) If a non-increasing sequence $fy_n \to b$, then $fy_n \geq b$ for all n,
(iii) If a non-decreasing sequence $fz_n \to c$, then $fz_n \leq c$ for all n, and
(iv) If a non-increasing sequence $fw_n \to d$, then $fw_n \geq d$ for all n.
If there exist $x_0, y_0, z_0, w_0 \in X$ with $fx_0 \leq A(x_0,y_0,z_0,w_0),\ fy_0 \geq A(y_0,z_0,w_0,x_0),$ $fz_0 \leq A(z_0,w_0,x_0,y_0),\ fw_0 \geq A(w_0,x_0,y_0,z_0)$ then A and f have a quadruple coincidence point in X.

Proof: $(x_0,y_0,z_0,w_0)\in X$ be such that $\begin{array}{l} fx_0 \leq A(x_0,y_0,z_0,w_0) = fx_1,\ fy_0 \geq A(y_0,z_0,w_0,x_0) = fy_1 \\ fz_0 \leq A(z_0,w_0,x_0,y_0) = fz_1,\ fw_0 \geq A(w_0,x_0,y_0,z_0) = fw_1 \end{array}$

where $(x_1,y_1,z_1,w_1)\in X$ Thus
$fx_0 \leq fx_1,\ fy_0 \geq fy_1, fz_0 \leq fz_1,\ f\,w_0 \geq fw_1$ Again
$x_2 = A(x_1,y_1,z_1,w_1),\ y_2 = A(y_1,z_1,w_1,x_1), z_2 = A(z_1,w_1,x_1,y_1), w_2 = A(w_1,x_1,y_1,z_1)$ $\because$ A has the mixed monotone property $x_0 \leq x_1 \leq x_2,\ y_0 \geq y_1 \geq y_2,\ z_0 \leq z_1 \leq z_2,\ w_0 \geq w_1 \geq w_2$

By continuing this process , construct the sequence $\{fx_n\},\{fy_n\},\{fz_n\},\{fw_n\}$ in X such that
$fx_{n+1} = A(x_n,y_n,z_n,w_n),\ fy_{n+1} = A(y_n,z_n,w_n,x_n), fz_{n+1} = A(z_n,w_n,x_n,y_n),$
$fw_{n+1} = A(w_n,x_n,y_n,z_n)$
Let us show that
$fx_n \leq fx_{n+1},\ fy_n \geq fy_{n+1}, fz_n \leq fz_{n+1},\ fw_n \geq fw_{n+1}$ ....(2)
From mathematical induction $fx_0 \leq A(x_0,y_0,z_0,w_0) = fx_1,\ fy_0 \geq A(y_0,z_0,w_0,x_0) = fy_1$
$fz_0 \leq A(z_0,w_0,x_0,y_0) = fz_1,\ fw_0 \geq A(w_0,x_0,y_0,z_0) = fw_1$ and
thus $fx_0 \leq fx_1,\ fy_0 \geq fy_1, fz_0 \leq fz_1,\ f\,w_0 \geq fw_1$), thus (2) holds for n=0.
Let us presume that (2) holds for n > 0. As A has the mixed g-monotone property and
$fx_n \leq fx_{n+1},\ fy_n \geq fy_{n+1}, fz_n \leq fz_{n+1},\ fw_n \geq fw_{n+1},$
we obtain

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$$fx_{n+1} = A(x_n, y_n, z_n, w_n)$$
$$< A(x_{n+1}, y_n, z_n, w_n) < A(x_{n+1}, y_n, z_{n+1}, w_n)$$
$$< A(x_{n+1}, y_{n+1}, z_{n+1}, w_n) < A(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) = fx_{n+2}$$
$$\therefore fx_{n+1} \le fx_{n+2}$$
$$fy_{n+2} = A(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) < A(y_{n+1}, z_n, w_{n+1}, x_{n+1})$$
$$< A(y_n, z_n, w_{n+1}, x_{n+1}) < A(y_n, z_n, w_{n+1}, x_n)$$
$$< A(y_n, z_n, w_n, x_n) = fy_{n+1}$$
$$\therefore fy_{n+2} < fy_{n+1}$$
$$fz_{n+1} = A(z_n, w_n, x_n, y_n) < A(z_{n+1}, w_n, x_n, y_n)$$
$$< A(z_{n+1}, w_{n+1}, x_n, y_n) < A(z_{n+1}, w_{n+1}, x_{n+1}, y_n)$$
$$< A(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}) = fz_{n+2}$$
$$\therefore fz_{n+1} < fz_{n+2}$$
$$fw_{n+2} = A(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) < A(w_{n+1}, x_n, y_{n+1}, z_{n+1})$$
$$< A(w_n, x_n, y_{n+1}, z_{n+1}) < A(w_n, x_n, y_n, z_{n+1})$$
$$< A(w_n, x_n, y_n, z_n) = fw_{n+1}$$
$$\therefore fw_{n+2} < fw_{n+1}$$

Thus, (2) holds for any $n \in N$. Assume for some $n \in N$,

$$fx_n = fx_{n+1},\ fy_n = fy_{n+1}, fz_n = fz_{n+1},\ fw_n = fw_{n+1}$$

Thus $fx_n = A(x_n, y_n, z_n, w_n), fy_n = A(y_n, z_n, w_n, x_n),\ fz_n = A(z_n, w_n, x_n, y_n)$,

$fw_n = A(w_n, x_n, y_n, z_n) \Rightarrow (x_n, y_n, z_n, w_n)$ is a quadruple coincidence point of A and f.

Now, for any $n \in N$, $fx_n \ne fx_{n+1},\ fy_n \ne fy_{n+1}, fz_n \ne fz_{n+1},\ fw_n \ne fw_{n+1}$.

Putting (x, y, z, w) = (x_n, y_n, z_n, w_n) and (p, q, r, s) = $(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1})$

$$H\begin{pmatrix} \left(d(A(w_n, x_n, y_n, z_n), A(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1})\right), \\ \left(d(A(z_n, w_n, x_n, y_n), A(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1})\right), \\ \left(d(A(y_n, z_n, w_n, x_n), A(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1})\right), \\ \left(d(A(x_n, y_n, z_n, w_n), A(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1})\right), \\ \left(\left(d(A(x_n, y_n, z_n, w_n), f(x_n)\right) + \left(d(A(x_{n-1}, q_{n-1}, r_{n-1}, s_{n-1}), f(x_{n-1})\right)\right), \\ \left(d\left(f(x_n), f(x_{n-1})\right)\right), \left(d\left(f(y_n), f(q_{n-1})\right)\right), \left(d\left(f(z_n), f(z_{n-1})\right)\right), \\ \left(d\left(f(w_n), f(w_{n-1})\right)\right), \\ \left(\left(d(A(x_n, y_n, z_n, w_n), f(x_{n-1})\right) + \left(d(A(x_{n-1}, q_{n-1}, r_{n-1}, s_{n-1}), f(x_n)\right)\right) \end{pmatrix} \le 0$$

$$\therefore H\begin{pmatrix} \left(d(fw_{n+1},fw_n)\right),\left(d(fz_{n+1},fz_n)\right),\left(d(fy_{n+1},fy_n)\right),\left(d(fx_{n+1},fx_n)\right), \\ \left(\left(d\left(fx_{n+1},fx_n\right)\right)+\left(d\left(fx_n,fx_{n-1}\right)\right)\right),\left(d\left(fx_n,fx_{n-1}\right)\right), \\ \left(d\left(fy_n,fy_{n-1}\right)\right),\left(d\left(fz_n,fz_{n-1}\right)\right),\left(d\left(fw_n,fw_{n-1}\right)\right), \\ \left(\left(d(fx_{n+1},fx_{n-1}\right)+\left(d(fx_n,fx_n)\right)\right) \end{pmatrix}\le 0$$

By triangle inequality ,

$$\therefore H\begin{pmatrix} \left(d(fw_{n+1},fw_n)\right),\left(d(fz_{n+1},fz_n)\right),\left(d(fy_{n+1},fy_n)\right),\left(d(fx_{n+1},fx_n)\right), \\ \left(\left(d\left(fx_{n+1},fx_n\right)\right)+\left(d\left(fx_n,fx_{n-1}\right)\right)\right),\left(d\left(fx_n,fx_{n-1}\right)\right), \\ \left(d\left(fy_n,fy_{n-1}\right)\right),\left(d\left(fz_n,fz_{n-1}\right)\right),\left(d\left(fw_n,fw_{n-1}\right)\right), \\ \left(\left(d(fx_{n+1},fx_n\right)+\left(d(fx_n,fx_{n-1})\right)\right) \end{pmatrix}\le 0$$

Thus by (H2), we get

$$\left(d(fw_{n+1},fw_n)+d(fz_{n+1},fz_n)+d(fy_{n+1},fy_n)+d(fx_{n+1},fx_n)\right)$$
$$< h\left(d\left(fx_n,fx_{n-1}\right)+d\left(fy_n,fy_{n-1}\right)+d\left(fz_n,fz_{n-1}\right)+d\left(fw_n,fw_{n-1}\right)\right)$$

Thus we may show that

$$\left(d(fw_{n+1},fw_n)+d(fz_{n+1},fz_n)+d(fy_{n+1},fy_n)+d(fx_{n+1},fx_n)\right)$$
$$< h\left(d\left(fx_n,fx_{n-1}\right)+d\left(fy_n,fy_{n-1}\right)+d\left(fz_n,fz_{n-1}\right)+d\left(fw_n,fw_{n-1}\right)\right)$$
$$< h^2\left(d\left(fx_{n-1},fx_{n-2}\right)+d\left(fy_{n-1},fy_{n-2}\right)+d\left(fz_{n-1},fz_{n-2}\right)+d\left(fw_{n-1},fw_{n-2}\right)\right)$$

for all n=0,

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$$< h^n\left(d\left(fx_1,fx_0\right)+d\left(fy_1,fy_0\right)+d\left(fz_1,fz_0\right)+d\left(fw_1,fw_0\right)\right)$$

1,2...........

Let us show that $\{fx_n\},\{fy_n\},\{fz_n\},\{fw_n\}$ are Cauchy sequences, let m>n.

Then, by the triangle inequality, we have

$$\left(d(fw_m,fw_n)+d(fz_m,fz_n)+d(fy_m,fy_n)+d(fx_m,fx_n)\right)$$
$$< \sum_{i=n}^{m} h^i\left(d\left(fx_1,fx_0\right)+d\left(fy_1,fy_0\right)+d\left(fz_1,fz_0\right)+d\left(fw_1,fw_0\right)\right)$$
$$< \left(\sum_{i=0}^{m} h^i-\sum_{i=0}^{n-1} h^i\right)\left(d\left(fx_1,fx_0\right)+d\left(fy_1,fy_0\right)+d\left(fz_1,fz_0\right)+d\left(fw_1,fw_0\right)\right)$$

Taking $m,n\to\infty$, we get $\lim_{m,n\to\infty}\left(d(fw_m,fw_n)+d(fz_m,fz_n)+d(fy_m,fy_n)+d(fx_m,fx_n)\right)=0$

$\lim_{m,n\to\infty} d(fw_m,fw_n)=0,\ \lim_{m,n\to\infty} d(fz_m,fz_n)=0,\ \lim_{m,n\to\infty} d(fy_m,fy_n)=0,\ \lim_{m,n\to\infty} d(fx_m,fx_n)=0$. Hence the sequences $\{fx_n\},\{fy_n\},\{fz_n\},\{fw_n\}$ are Cauchy sequences.

X is a complete metric space, there exists $a,b,c,d\in X$ such that

$\lim_{n\to\infty} fx_n = a$; $\lim_{n\to\infty} fy_n = b$; $\lim_{n\to\infty} fz_n = c$; $\lim_{n\to\infty} fw_n = d$ ....(2)

Thus we get $\lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} A(x_n, y_n, z_n, w_n) = a$; $\lim_{n\to\infty} fy_{n+1} = \lim_{n\to\infty} A(y_n, z_n, w_n, x_n) = b$;
$\lim_{n\to\infty} fz_{n+1} = \lim_{n\to\infty} A(z_n, w_n, x_n, y_n) = c$; $\lim_{n\to\infty} fw_{n+1} = \lim_{n\to\infty} A(w_n, x_n, y_n, z_n) = d$

A, f are O-compatible, we get

$$\lim_{n\to\infty} fA(x_n, y_n, z_n, w_n) = \lim_{n\to\infty} A(fx_n, fy_n, fz_n, fw_n);$$
$$\lim_{n\to\infty} fA(y_n, z_n, w_n, x_n) = \lim_{n\to\infty} A(fy_n, fz_n, fw_n, fx_n);$$
$$\lim_{n\to\infty} fA(z_n, w_n, x_n, y_n) = \lim_{n\to\infty} A(fz_n, fw_n, fx_n, fy_n); \quad \text{..(3)}$$
$$\lim_{n\to\infty} fA(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} A(fw_n, fx_n, fy_n, fz_n)$$

Now by continuity of A (i.e. (a)), we get

$$d(fa, A(fx_n, fy_n, fz_n, fw_n)) \le d(fa, fA(x_n, y_n, z_n, w_n)) + d(fA(x_n, y_n, z_n, w_n), A(fx_n, fy_n, fz_n, fw_n))$$

Taking $\lim_{n\to\infty}$,we get

$$d(fa, A(a,b,c,d)) \le 0 \Rightarrow fa = A(a,b,c,d)$$

Similarly we may show that

$$d(fb, A(b,c,d,a)) \le 0 \Rightarrow fb = A(b,c,d,a); d(fc, A(c,d,a,b)) \le 0 \Rightarrow fc = A(c,d,a,b);$$
$$d(fd, A(d,a,b,c)) \le 0 \Rightarrow fd = A(d,a,b,c)$$

Thus (a, b, c, d) is a quadruple coincidence point of A, f.
Now assume that (b) holds.

(i) If a non-decreasing sequence $fx_n \to a$, then $fx_n \le a$ for all n,

(ii) If a non-increasing sequence $fy_n \to b$, then $fy_n \ge b$ for all n,

(iii) If a non-decreasing sequence $fz_n \to c$, then $fz_n \le c$ for all n, and

(iv) If a non-increasing sequence $fw_n \to fw$, then $fw_n \ge fw$ for all n.

Since $\{fx_n\}$ is a non-decreasing sequence and $fx_n \to a$, then $fx_n \le a$,

Then we have $ffx_n \to fa$ for all for all n $\in$ N by (i).

Similarly, since $\{fy_n\}$ is a non-increasing sequence and $fy_n \to b$, we also have $ffy_n \to fb$ for all n $\in$ N.

Similarly $ffz_n \to fc$ and $ffw_n \to fd$.

Since f is continuous,

$$\lim_{n\to\infty} d(ffx_n, fa) = \lim_{n\to\infty} d(fa, fa) = \lim_{n\to\infty} d(fA(x_n, y_n, z_n, w_n), fa) = \lim_{n\to\infty} d(A(fx_n, fy_n, fz_n, fw_n), fa)$$
$$\lim_{n\to\infty} d(ffy_n, fb) = \lim_{n\to\infty} d(fb, fb) = \lim_{n\to\infty} d(fA(y_n, z_n, w_n, x_n), fb) = \lim_{n\to\infty} d(A(fy_n, fz_n, fw_n, fx_n), fb)$$
$$\lim_{n\to\infty} d(ffz_n, fc) = \lim_{n\to\infty} d(fc, fc) = \lim_{n\to\infty} d(fA(z_n, w_n, x_n, y_n), fc) = \lim_{n\to\infty} d(A(fz_n, fw_n, fx_n, fy_n), fc)$$
$$\lim_{n\to\infty} d(ffw_n, fd) = \lim_{n\to\infty} d(fd, fd) = \lim_{n\to\infty} d(fA(w_n, x_n, y_n, z_n), fd) = \lim_{n\to\infty} d(A(fw_n, fx_n, fy_n, fz_n), fd)$$

Putting (x, y, z, w) = (fx_n, fy_n, fz_n, fw_n) and (p, q, r, s) = (a,b,c,d) in (1), we get

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$$H\begin{pmatrix} (d(A(fw_n,fx_n,fy_n,fz_n),A(d,a,b,c)),(d(A(fz_n,fw_n,fx_n,fy_n),A(c,d,a,b)), \\ (d(A(fy_n,fz_n,fw_n,fx_n),A(b,c,d,a)),(d(A(fx_n,fy_n,fz_n,fw_n),A(a,b,c,d)), \\ ((d(A(fx_n,fy_n,fz_n,fw_n),f(fx_n))+(d(A(a,b,c,d),f(a))),(d(ffx_n,fa)),(d(ffy_n,fb)), \\ (d(ffz_n,fc)),(d(ffw_n,fd)),((d(A(fx_n,fy_n,fz_n,fw_n),f(a))+(d(A(a,b,c,d),ffx_n)) \end{pmatrix}\le 0$$ Taking $\lim_{n\to\infty}$,we get

$$H\begin{pmatrix} d(fd,A(d,a,b,c),d(fc,A(c,d,a,b),d(fb,A(b,c,d,a),(d(fa,A(a,b,c,d)), \\ d(A(a,b,c,d),f(a)),0,0,0,0,d(A(a,b,c,d),fa) \end{pmatrix}\le 0$$

From (H2), we get

$$d(fd,A(d,a,b,c)+d(fc,A(c,d,a,b)+d(fb,A(b,c,d,a)+(d(fa,A(a,b,c,d))\le h(0+0+0+0)$$

$$d(fd,A(d,a,b,c))=0\Rightarrow fd=A(d,a,b,c)$$

$$d(fc,A(c,d,a,b))=0\Rightarrow fc=A(c,d,a,b)$$

$$d(fb,A(b,c,d,a))=0\Rightarrow fb=A(b,c,d,a)$$

$$d(fa,A(a,b,c,d))=0\Rightarrow fa=A(a,b,c,d)$$

Thus (a, b, c, d) is a quadruple coincidence point of A, f.

COROLLARY-1:

This is the extension of main result of Lakshmikantham and Ciric [19] from coupled fixed point to quadruple coincidence point theorem..

Let (X, d, $\le$) be a partially ordered complete metric space. Suppose that $A:X\times X\times X\times X\to X$ and f: X →X are two mappings such that A has the mixed g-monotone property. Assume that there exists H $\in \Upsilon$ such that

(i) $d(A(x,y,z,w),A(p,q,r,s)\le \xi\left(\frac{1}{4}\left(d(fx,fp)+d(fy,fq)+d(fz,fr)+d(fw,fs)\right)\right)$

for all x, y, z, w, p, q, r, s $\in$ X with $f(x)\ge f(p),\ f(y)\le f(q),\ f(z)\ge f(r),\ f(w)\le f(s)$. Also $A(X\times X\times X\times X)\subseteq f(x)$ and g is continuous on X and O-compatible with A. Additionally, suppose that either

(a) F is continuous, or

(b) X has the properties

(i) If a non-decreasing sequence $fx_n\to a$, then $fx_n\le a$ for all n,

(ii) If a non-increasing sequence $fy_n\to b$, then $fy_n\ge b$ for all n,

(iii) If a non-decreasing sequence $fz_n\to c$, then $fz_n\le c$ for all n, and

(iv) If a non-increasing sequence $fw_n\to d$, then $fw_n\ge d$ for all n.

If there exist $x_0,y_0,z_0,w_0\in X$ with $fx_0\le A(x_0,y_0,z_0,w_0),\ fy_0\ge A(y_0,z_0,w_0,x_0),$

$fz_0\le A(z_0,w_0,x_0,y_0),\ fw_0\ge A(w_0,x_0,y_0,z_0)$ then A and f have a quadruple coincidence point in X.

Proof: By using Example -4, we get the result.

Corrollary-2: This is the extension of main result of Bhaskar and Lakshmikantham [17] from coupled fixed point to quadruple coincidence point theorem.

Let (X, d, $\le$) be a partially ordered complete metric space. Suppose that $A:X\times X\times X\times X\to X$ and f: X →X are two mappings such that A has the mixed g-monotone property.

(i) $d(A(x,y,z,w),A(p,q,r,s)\le \frac{\kappa}{4}\left(\left(d(fx,fp)+d(fy,fq)+d(fz,fr)+d(fw,fs)\right)\right)$

for all x, y, z, w, p, q, r, s $\in$ X with $f(x)\ge f(p),\ f(y)\le f(q),\ f(z)\ge f(r),\ f(w)\le f(s)$ and $\kappa\in(0,1)$. Also $A(X\times X\times X\times X)\subseteq f(x)$ and g is continuous on X and O-compatible with A. Additionally, suppose that either

(a) A is continuous, or

(b) X has the properties

(i) If a non-decreasing sequence $fx_n \to a$, then $fx_n \le a$ for all n,

(ii) If a non-increasing sequence $fy_n \to b$, then $fy_n \ge b$ for all n,

(iii) If a non-decreasing sequence $fz_n \to c$, then $fz_n \le c$ for all n, and

(iv) If a non-increasing sequence $fw_n \to d$, then $fw_n \ge d$ for all n.

If there exist $x_0, y_0, z_0, w_0 \in X$ with $fx_0 \le A(x_0, y_0, z_0, w_0),\ fy_0 \ge A(y_0, z_0, w_0, x_0),$ $fz_0 \le A(z_0, w_0, x_0, y_0),\ fw_0 \ge A(w_0, x_0, y_0, z_0)$ then A and f have a quadruple coincidence point in X.

Proof: By using Example-2 we get the result.

Corrollary-3: This is the extension of main result of Abbas et.al.[12] from coupled fixed point to quadruple coincidence point theorem.

Let (X, d, $\le$) be a partially ordered complete metric space. Suppose that $A: X \times X \times X \times X \to X$ and f: X →X are two mappings such that A has the mixed g-monotone property.

(i)
$$d(A(x,y,z,w),A(p,q,r,s)) \le \alpha\Big(\big(d(A(x,y,z,w),f(x)\big)+\big(d(A(p,q,r,s),f(p))\big)\Big)+\beta d\big(fx,fp\big)$$
$$+\delta d\big(fy,fq\big)+\theta d\big(fz,fr\big)+\omega d(fw,fs)+\lambda\Big(\big(d(A(x,y,z,w),f(p)\big)+\big(d(A(p,q,r,s),f(x))\big)\Big)$$

for all x, y, z, w, p, q, r, s $\in X$ with $f(x) \ge f(p),\ f(y) \le f(q),\ f(z) \ge f(r),\ f(w) \le f(s)$ and $\kappa \in (0,1)$. Also $A(X \times X \times X \times X) \subseteq f(x)$ and g is continuous on X and O-compatible with A. Additionally, suppose that either

(a) A is continuous, or

(b) X has the properties

(i) If a non-decreasing sequence $fx_n \to a$, then $fx_n \le a$ for all n,

(ii) If a non-increasing sequence $fy_n \to b$, then $fy_n \ge b$ for all n,

(iii) If a non-decreasing sequence $fz_n \to c$, then $fz_n \le c$ for all n, and

(iv) If a non-increasing sequence $fw_n \to d$, then $fw_n \ge d$ for all n.

If there exist $x_0, y_0, z_0, w_0 \in X$ with $fx_0 \le A(x_0, y_0, z_0, w_0),\ fy_0 \ge A(y_0, z_0, w_0, x_0),$ $fz_0 \le A(z_0, w_0, x_0, y_0),\ fw_0 \ge A(w_0, x_0, y_0, z_0)$ then A and f have a quadruple coincidence point in X.

Proof: By using example -1 ,we get the result.

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