Fitted Finite Difference Method for Singularly Perturbed Two-Point Boundary Value Problems using Polynomial Cubic Spline

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Abstract— In this paper, a fitted finite difference method using polynomial cubic spline on uniform mesh is presented for solving singularly perturbed two-point boundary value problems with boundary layer at one end (left or right) point. A fitting factor called artificial viscosity is introduced in a tridiagonal finite difference scheme and its value is obtained from the theory of asymptotic solution singular perturbations. We have solved the tridiagonal scheme obtained by the method using discrete invariant imbedding. To demonstrate the applicability of the method, we have solved several model examples. Maximum absolute errors in the numerical solution of the problems are presented to illustrate and justify the method.

Keywords- *Singularly perturbed boundary value problem, Fitting factor, Cubic spline, Tridiagonal system, Invariant imbedding algorithm*

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I. INTRODUCTION

Singularly perturbed boundary value problems are widespread in nature. Typically these problems in various fields of applied mathematics such as fluid mechanics, elasticity, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction diffusion process, geophysics and many other areas. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution derivatives are extremely large. The numerical treatment of singularly perturbed differential equations gives major computational difficulties due to the presence of boundary and/or interior layers. A wide verity of papers and books have been published in the recent years, describing various methods for solving singularly perturbed two-point boundary value problems, among these, we mention [1-13].

In this paper, an exponentially fitted finite difference scheme using polynomial cubic spline on an uniform mesh is presented for solving singularly perturbed two-point boundary value problems with boundary layer at one end (left or right) point. In section 2, we define the polynomial spline and in section 3, fitted finite difference method using spline is presented. To illustrate the proposed method, we have considered numerical examples in section 4. Finally, we have given discussions and conclusion to justify the proposed method.

II. POLYNOMIAL SPLINE METHOD

We divide the interval [0, 1] into N equal subintervals of mesh size $h = \frac{1}{N}$, so that $x_i = ih,\ i = 0, 1, 2, \ldots, N$ with $0 = x_0, 1 = x_N$. Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$ obtained by the polynomial cubic spline $S_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). We do not only require that $S_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1}, but also the continuity of first derivative at the common nodes (x_i, y_i) are fulfilled.

For each $i\,th$ segment, the cubic polynomial spline function $S_i(x)$ has the form

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,\ i = 0, 1, \ldots, N-1. \quad (1)$$

where a_i, b_i, c_i and d_i are constants.

A polynomial function $S(x)$ of class $C^2[a,b]$ interpolates $y(x)$ at the grid points x_i for i = 0,1,2.......,N. To derive an expression for the coefficients of Eq. (1) in term of y_i, y_{i+1}, M_i and M_{i+1}, we first define

$$S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1},$$
$$S_i''(x_i) = M_i, S_i''(x_{i+1}) = M_{i+1}$$

From algebraic manipulation, we get the following expression:

$$a_i = y_i, b_i = \frac{y_{i+1} - y_i}{h} - \frac{h(M_{i+1} + 2M_i)}{6},$$
$$c_i = \frac{M_i}{2}, d_i = \frac{M_{i+1} - M_i}{6h}$$

where i = 0, 1, 2... N-1.

Using the continuity of the first derivative at (x_i, y_i), that is $S_{i-1}'(x_i) = S_i'(x_i)$, we obtain the following relations for i =1,2,...,N-1.

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) \quad (2)$$

III. EXPONENTIAL FITTED CUBIC SPLINE FINITE DIFFERNCE METHOD

To describe the method, we first consider singularly perturbed two point boundary value problems of the form

$$\varepsilon y''(x)+p(x)y'(x)+q(x)y(x)=r(x), \quad 0\le x\le 1, \qquad (3)$$

with boundary conditions $y(0)=\alpha$ (4a)

and $y(1)=\beta$ (4b)

where $0<\varepsilon<<1$, $p(x)$, $q(x)$ and $r(x)$ are bounded continuous functions in (0, 1), and α,β are finite constants.

Left – end boundary layer

Assume that $q(x)\le 0, p(x)\ge \overline{M}>0$ throughout the interval [0, 1], where $\overline{M}$ is positive constant, then the boundary layer will be in the neighbourhood of $x = 0$ for small values of ε

From the theory of singular perturbation it is known that the solution of (3) and (4) is of the form (O'Malley [7], pp.22-26)

$$y(x)=y_0(x)+\frac{p(0)}{p(x)}(\alpha-y_0(0))e^{-\int_0^x\left(\frac{p(x)}{\varepsilon}-\frac{q(x)}{p(x)}\right)dx}+O(\varepsilon) \qquad (5)$$

where $y_0(x)$ is the solution of reduced problem $p(x)y_0'(x)+q(x)y_0(x)=r(x)$, with $y_0(1)=\beta$.

By taking the Taylor's series expansion for $p(x)$ and $q(x)$ about the point '0' and restricting to their first terms, (5) becomes

$$y(x)=y_0(x)+(\alpha-y_0(0))e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)x}+O(\varepsilon) \qquad (6)$$

We divide the interval [0, 1] into N equal subintervals of mesh size $h=\frac{1}{N}$, so that $x_i=ih,\ i=0,1,\ldots,N$ with $0=x_0,\ 1=x_N$.

From (6) we have

$$y(x_i)=y_0(x_i)+(\alpha-y_0(0))e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)x_i}+O(\varepsilon),$$

$$\text{i.e., } y(ih)=y_0(ih)+(\alpha-y_0(0))e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)ih}+O(\varepsilon),$$

$$\therefore \lim_{h\to 0} y(ih)=y_0(0)+(\alpha-y_0(0))e^{-\left(\frac{p^2(0)-\varepsilon q(0)}{p(0)}\right)i\rho}+O(\varepsilon) \qquad (7)$$

where $\rho=\frac{h}{\varepsilon}$

At the grid point x_i, the proposed differential equation (3) may be discretized by

$$\varepsilon M_j+p(x_j)y_j'(x)+q(x_j)y(x_j)=r(x_j) \text{ for } j=i,i\pm 1$$

Substituting the above equations in equation (2) and using the following approximations for the first order derivative of y

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h}+O(h^2)$$

$$y_{i+1}'=\frac{3y_{i+1}-4y_i+y_{i-1}}{2h}+O(h^2) \qquad (8)$$

$$y_{i-1}'=\frac{-y_{i+1}+4y_i-3y_{i-1}}{2h}+O(h^2)$$

We get

$$\frac{6\varepsilon}{h^2}(y_{i+1}-2y_i+y_{i-1})= \left(\frac{-p_{i+1}}{2h}+\frac{2p_i}{h}-q_{i-1}+\frac{3p_{i-1}}{2h}\right)y_{i-1} +\left(\frac{2p_{i+1}}{h}-4q_{i-1}-\frac{2p_{i-1}}{h}\right)y_i +\left(\frac{-3p_{i+1}}{2h}-\frac{2p_i}{h}-q_{i+1}+\frac{p_{i-1}}{2h}\right)y_{i+1} +(r_{i+1}+4r_i+r_{i-1}) \qquad (9)$$

Introducing fitting factor $\sigma(\rho)$ in Eq. (9), we get

$$\frac{6\sigma(\rho)\varepsilon}{h^2}(y_{i+1}-2y_i+y_{i-1})= \left(\frac{-p_{i+1}}{2h}+\frac{2p_i}{h}-q_{i-1}+\frac{3p_{i-1}}{2h}\right)y_{i-1} +\left(\frac{2p_{i+1}}{h}-4q_{i-1}-\frac{2p_{i-1}}{h}\right)y_i +\left(\frac{-3p_{i+1}}{2h}-\frac{2p_i}{h}-q_{i+1}+\frac{p_{i-1}}{2h}\right)y_{i+1} +(r_{i+1}+4r_i+r_{i-1}) \qquad (10)$$

Multiplying (10) by h and taking $h\to 0$, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left(y(i+1)h-2y(ih)+y(i-1)h\right) = \frac{p(0)}{2}\lim_{h\to 0}\left(y(i-1)h-y(i+1)h\right). \qquad (11)$$

By substituting (7) in to (11) we get

$$\sigma=\frac{\rho}{2}p(0)\coth\left(\left(\frac{p(0)^2-\varepsilon q(0)}{p(0)}\right)\frac{\rho}{2}\right) \qquad (12)$$

is the constant fitting factor.

From Eq.(10) we get the following tridiagonal system

$$E_i y_{i-1}+F_i\, y_i+G_i y_{i+1}=H_i \quad \text{for } i=1,2,\ldots,N-1 \qquad (13)$$

where

$$E_i=\frac{6\sigma}{\rho}-\frac{3}{2}p_{i-1}-2p_i+\frac{1}{2}p_{i+1}+h\,q_{i-1},$$

$$F_i=\frac{-12\sigma}{\rho}+2\,p_{i-1}-2\,p_{i+1}+4h\,q_i$$

$$G_i=\frac{6\sigma}{\rho}-\frac{1}{2}p_{i-1}+2p_i+\frac{3}{2}p_{i+1}+h\,q_{i+1},$$

$$p(x_i)=p_i,\ q(x_i)=q_i,\ r(x_i)=r_i,\quad \rho=\frac{h}{\varepsilon} \quad \text{for } i=0,1,\ldots,N$$

We solve the tridiagonal system (13), using method of invariant imbedding algorithm.

Right-end boundary layer

Further, if we assume that $p(x)\le \overline{M}<0$ throughout the interval [0, 1] where $\overline{M}$ is negative constant, then the boundary layer will be in the neighbourhood of $x = 1$.

From the theory of singular perturbation it is known that the solution of (3) and (4) is of the form

$$y(x)=y_0(x)+\frac{p(1)}{p(x)}(\beta-y_0(1))e^{\int_0^x\left(\frac{p(x)}{\varepsilon}-\frac{q(x)}{p(x)}\right)dx}+O(\varepsilon) \qquad (14)$$

where $y_0(x)$ is the solution of

$p(x)y_0'(x)+q(x)y_0(x)=r(x)$, with $y_0(0)=\alpha$.

By taking the Taylor's series expansion for $p(x)$ and $q(x)$ about the point '1' and restricting to their first terms, (14) becomes

$$y(x)=y_0(x)+(\beta-y_0(1))e^{\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(1-x)}+O(\varepsilon) \qquad (15)$$

$$y(x_i)=y_0(x_i)+(\beta-y_0(1))e^{\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(1-x_i)}+O(\varepsilon),$$

i.e., $y(ih)=y_0(ih)+(\beta-y_0(1))e^{\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(1-ih)}+O(\varepsilon)$,

$$\therefore \lim_{h\to 0} y(ih)=y_0(0)+(\beta-y_0(1))e^{\left(\frac{p^2(1)-\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon}-i\rho\right)}+O(\varepsilon),$$

where $\rho=\frac{h}{\varepsilon}$ (16)

Now, we consider cubic spline finite difference scheme and introduce the fitting factor $\sigma(\rho)$ as:

$$\frac{6\sigma(\rho)\varepsilon}{h^2}(y_{i+1}-2y_i+y_{i-1})=\left(\frac{-p_{i+1}}{2h}+\frac{2p_i}{h}-q_{i-1}+\frac{3p_{i-1}}{2h}\right)y_{i-1}+\left(\frac{2p_{i+1}}{h}-4q_{i-1}-\frac{2p_{i-1}}{h}\right)y_i+\left(\frac{-3p_{i+1}}{2h}-\frac{2p_i}{h}-q_{i+1}+\frac{p_{i-1}}{2h}\right)y_{i+1}+(r_{i+1}+4r_i+r_{i-1}) \qquad (17)$$

Multiplying (17) by h and taking limit as $h\to 0$, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho}(y(i+1)h-2y(ih)+y(i-1)h)=\frac{p(0)}{2}\lim_{h\to 0}(y(i-1)h-y(i+1)h). \qquad (18)$$

By substituting (16) in to (18) we get

$$\sigma=\frac{\rho}{2}p(0)\coth\left(\left(\frac{p(1)^2-\varepsilon q(1)}{p(1)}\right)\frac{\rho}{2}\right) \qquad (19)$$

is the constant fitting factor in the right end boundary layer.

From Eq.(17) we get the following tridiagonal system

$$E_i y_{i-1}+F_i y_i+G_i y_{i+1}=H_i \quad \text{for } i=1,2,\ldots,N-1 \qquad (20)$$

where

$$E_i=\frac{6\sigma}{\rho}-\frac{3}{2}p_{i-1}-2p_i+\frac{1}{2}p_{i+1}+h\,q_{i-1},$$

$$F_i=\frac{-12\sigma}{\rho}+2p_{i-1}-2p_{i+1}+4h\,q_i$$

$$G_i=\frac{6\sigma}{\rho}-\frac{1}{2}p_{i-1}+2p_i+\frac{3}{2}p_{i+1}+h\,q_{i+1},$$

$$H_i=h[r_{i-1}+4r_i+r_{i+1}]$$

$$p(x_i)=p_i,\ q(x_i)=q_i,\ r(x_i)=r_i,\quad \rho=\frac{h}{\varepsilon}$$

for $i=0,1,\ldots,N$

To solve the tridiagonal system (20), we used method of invariant imbedding algorithm.

IV NUMERICAL EXAMPLES

To demonstrate the applicability of proposed method computationally, we consider two singularly perturbed two-point boundary value problems with left-end boundary layer and one problem with right end boundary layer of the underlying interval. These problems have been chosen because they have been widely discussed in the literature and because exact solutions are available for comparison.

Example 1. Consider the following homogeneous singular perturbation problem from Bender and Orszag[[1], page 480; problem 9.17 with α=0]

$\varepsilon y''(x)+y'(x)-y(x)=0;\ x\in[0,1]$ with $y(0)=1$ and $y(1)=1$.

Clearly this problem has a boundary layer at x = 0.

The exact solution is given by

$$y(x)=\frac{\left[(e^{m_2}-1)e^{m_1x}+(1-e^{m_1})e^{m_2x}\right]}{(e^{m_2}-e^{m_1})}$$

where $m_1=(-1+\sqrt{1+4\varepsilon})/2\varepsilon$ and $m_2=(-1-\sqrt{1+4\varepsilon})/2\varepsilon$.

The maximum absolute errors in the solution with and without fitting factor are presented in Table 1.

Example 2. Consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt[[8], example 2]

$\varepsilon y''(x)+y'(x)=1+2x;\ x\in[0,1]$ with $y(0)=0$ and $y(1)=1$.

Clearly this problem has a boundary layer at x = 0.The exact solution is given by

$$y(x)=x(x+1-2\varepsilon)+\frac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}.$$

The maximum absolute errors in the solution with and without fitting factor are presented in Table 2.

Example 3. Consider the following singular perturbation problem $\varepsilon y''(x)-y'(x)-(1+\varepsilon)y(x)=0;\quad x\in[0,1]$

with $y(0)=1+\exp(-(1+\varepsilon)/\varepsilon)$ and $y(1)=1+1/e$

Clearly this problem has a boundary layer at x = 1. The exact solution is given by $y(x)=e^{(1+\varepsilon)(x-1)/\varepsilon}+e^{-x}$

The maximum absolute errors in the solution with and without fitting factor are presented in Table 3.

ISSN: 2321-8169
056– 060

V DISCUSSIONS AND CONCLUSION

We have discussed a fitted finite difference method using cubic spline for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. We have introduced a fitting factor which is called artificial viscosity in the spline difference scheme, which control the rapid changes that occur in the boundary layer. To demonstrate the applicability of the method, we have tested it on three examples. Solutions of these problems using present method are compared with finite difference scheme with spline without fitting factor. From the numerical results, it is observed that the proposed method has better approximation to the exact solution and shows the importance of the fitting factor introduced in the finite difference scheme.

Table1. Maximum absolute errors in solution of Example 1

ε / N	2^4	2^5	2^6	2^7	2^8
With fitting factor					
10^{-3}	1.06(-2)	5.30(-3)	2.50(-3)	1.10(-3)	3.78(-4)
10^{-4}	1.09(-2)	5.60(-3)	2.80(-3)	1.40(-3)	6.79(-4)
10^{-6}	1.09(-2)	5.60(-3)	2.80(-3)	1.40(-3)	7.16(-4)
10^{-8}	1.09(-2)	5.60(-3)	2.80(-3)	1.40(-3)	7.16(-4)
Without fitting factor					
10^{-3}	7.11(-1)	5.55(-1)	4.85(-1)	3.73(-1)	2.16(-1)
10^{-4}	1.05(+0)	8.87(-1)	6.80(-1)	5.99(-1)	5.69(-1)
10^{-6}	1.20(+0)	1.21(+0)	1.19(+0)	1.12(+0)	9.52(-1)
10^{-8}	1.20(+0)	1.22(+0)	1.22(+0)	1.23(+0)	1.22(+0)

Table 2. Maximum absolute errors in solution of Example 2

ε / N	2^4	2^5	2^6	2^7	2^8
With fitting factor					
10^{-3}	5.67(-2)	2.83(-2)	1.34(-2)	5.77(-3)	2.05(-3)
10^{-4}	5.84(-2)	3.01(-2)	1.52(-2)	7.55(-3)	3.69(-3)
10^{-6}	5.86(-2)	3.03(-2)	1.54(-2)	7.75(-3)	3.89(-3)
10^{-8}	5.86(-2)	3.03(-2)	1.54(-2)	7.75(-3)	3.89(-3)
Without fitting factor					
10^{-3}	2.02(+0)	9.09(-1)	7.71(-1)	5.91(-1)	3.42(-1)
10^{-4}	19.5(+0)	4.91(+0)	1.45(+0)	9.52(-1)	9.02(-1)
10^{-6}	1.95(+3)	4.88(+2)	1.22(+2)	30.52(+0)	7.66(+0)
10^{-8}	1.95(+5)	4.88(+4)	1.22(+4)	3.05(+3)	7.62(+2)

Table 3. Maximum absolute errors in Example 3

ε / N	2^4	2^5	2^6	2^7	2^8
With fitting factor					
10^{-3}	1.06(-2)	5.27(-3)	2.48(-3)	1.06(-3)	3.78(-4)
10^{-4}	1.09(-2)	5.59(-3)	2.81(-3)	1.39(-3)	6.80(-4)
10^{-6}	1.09(-2)	5.62(-3)	2.84(-3)	1.43(-3)	7.16(-4)
10^{-8}	1.09(-2)	5.62(-3)	2.84(-3)	1.43(-3)	7.16(-4)
Without fitting factor					
10^{-3}	1.13(+0)	8.79(-1)	7.69(-1)	5.91(-1)	3.42(-1)
10^{-4}	1.66(+0)	1.40(+0)	1.07(+0)	9.48(-1)	9.01(-1)
10^{-6}	1.90(+0)	1.91(+0)	1.89(+0)	1.77(+0)	1.51(+0)
10^{-8}	1.91(+0)	1.92(+0)	1.93(+0)	1.94(+0)	1.94(+0)

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