Application of Selection Rejection Methodology to some Statistical Distributions

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ABSTRACT

Abstract:- In this paper I have applied the Selection-Rejection Methodology to Nakagami-m distribution, Rayleigh distribution, Exponential distribution,Cauchy distribution and Beta distribution.

Keywords: - *random variable, iterations, target probability distribution and proposal probability distribution.*

I. INTRODUCTION.

The Selection-Rejection Methodology was developed based on the idea of Acceptance-Rejection Method by the renowned mathematician, Von Neumann, from the University of Berlin. Von Neumann[1] came forward with his method during 1950's but later on Karl Sigman[2] from Columbia University gave the similar methodology in 2007.Again in 1989,Bernard D.Flury[3] from Indiana University came forward with the theory "Acceptance-Rejection Sampling Made Easy".D.P.Kroese[4] from University of Queensland put forward his theory of Acceptance-Rejection in 2011.Selection-Rejection Methodology can be applied to almost all statistical distributions and hence it has got immense physical significance.

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II. SELECTION-REJECTION METHODOLOGY.

Let X be a continuous random variable with probability distribution function $f(x)\ \forall x \in R$,where R =set of all real numbers. Let $g(x)\ \forall x \in R$ where R =set of all real numbers be another probability density function such that $\frac{f(x)}{g(x)} \le k\ \ \forall\ x \in R$,where $k \ge 1$ is a real number. By successively selecting different values of X we will try to make the ratio $\frac{f(x)}{kg(x)}$ as close to 1 as possible. The probability density function $f(x)$ is called target distribution and the probability density function $g(x)$ is called proposal distribution.

The step by step procedure for the Selection-Rejection Methodology is as follows.

Step (1):- Let X be an one dimensional continuous random variable with probability distribution function $f(x)\ \forall x \in R$, where R =set of all real numbers.

Step (2):- Let X' be another one dimensional continuous random variable (which is independent of X) with probability distribution function $g(x)\ \forall x \in R$, where R =set of all real numbers.

Step (3):- Let $\frac{f(x)}{g(x)} \le k\ \ \forall\ x \in R$,where $k \ge 1$ a real number.

Step (4):- Let $0 < r < 1$ be a random number.

Step (5):- Set X' in terms of r depending on the expression obtained for the ratio $\frac{f(x)}{kg(x)}$.

Step (6):- If $r \le \frac{f(X')}{kg(X')}$, then set $X = X'$ and select the continuous random variable X'; otherwise reject the variable X' and repeat the process from step (1).

The probability that the continuous random variable X' is selected is $\frac{1}{k}$.

The number of iterations required to select X' is k.

It may be noted that $0 \le \frac{f(X')}{kg(X')} \le 1$.

Also X' is the candidate to be selected.

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III. APPLICATION TO NAKAGAMI-M DISTRIBUTION.

The Nakagami-m probability density function was proposed by Nakagami in 1960 as an empirical model for the amplitude of the received samples in wireless radio communications subject to multipath fading. Multipath fading is one of the most common distortions in wireless communications.

The Nakagami-m probability density function is given by

$$f(x)=\frac{2}{\Gamma m}\left(\frac{m}{\Omega}\right)^{m} x^{2m-1}\exp\left(-\frac{mx^{2}}{\Omega}\right) \quad (1)$$

Where Γm denotes the gamma function.

$\Omega=E\left(x^{2}\right)>0$, where $E\left(x^{2}\right)$ is the mathematical expectation.

$E\left(x^{2}\right)$ represents the average received power.

$m=\frac{\Omega^{2}}{Var(x^{2})}\geq 0.5$ is a fading parameter that characterizes the fading depth of the channel. The smaller the value of m,the higher is the fading depth.

Let $g(x)=x^{2m-1}$ be the proposal function. (2)

Here $f(x)$ is the target function.

$$\text{Let } h(x)=\frac{f(x)}{g(x)}=\frac{2}{\Gamma m}\left(\frac{m}{\Omega}\right)^{m}\exp\left(-\frac{mx^{2}}{\Omega}\right). \quad (3)$$

With the help of Differential Calculus it can be shown that $h(x)$ attains its maximum value at $x=0$.

The maximum value of $h(x)$ is $\frac{2}{\Gamma m}\left(\frac{m}{\Omega}\right)^{m}$.

Taking $K=\frac{2}{\Gamma m}\left(\frac{m}{\Omega}\right)^{m}$, we get

$$\frac{f(x)}{Kg(x)}=\exp\left(-\frac{mx^{2}}{\Omega}\right) \quad (4)$$

Now the steps for Selection-Rejection Methodology are as follows.

Step (1):- Let X be a continuous random variables with probability density function

$$f(x)=\frac{2}{\Gamma m}\left(\frac{m}{\Omega}\right)^{m} x^{2m-1}\exp\left(-\frac{mx^{2}}{\Omega}\right)$$

Step (2):- Let Y be another continuous random variable with probability density function $g(x)=x^{2m-1}$.

Step (3):- Let. $0<R<1$ be a random number.

Step (4):-Set $Y=\sqrt{-\frac{\Omega\ln(R)}{m}}$

Step (5):- If $R\leq\frac{f(Y)}{Kg(Y)}$,then set $X=Y$ and select the continuous random variable Y ;otherwise reject Y and repeat the process from step(1).

IV. APPLICATION TO RAYLEIGH DISTRIBUTION.

Rayleigh distribution is given by

$$f(x)=\frac{2(x-a)}{b}\exp\left[-\left(\frac{x-a}{b}\right)^2\right] \quad for \quad x\geq a \tag{5}$$

Let $g(x)=\frac{x-a}{b}$ be the prpoposal function. (6)

It may be noted that here $f\left(x\right)$ is the target function.

Then $$h(x)=2\exp\left[-\left(\frac{x-a}{b}\right)^2\right] \quad for \quad x\geq 0 \tag{7}$$

$h(x)$ attains its maximum value at $x=a$.

Maximum value of $h(x)$ is 2.

Taking $k=2$, we get

$$\frac{f(x)}{kg(x)}=\exp\left[-\left(\frac{x-a}{b}\right)^2\right] \tag{8}$$

Let $0<R_1<1$ be a random number.

Let $$\exp\left[-\left(\frac{x-a}{b}\right)^2\right]=R_1 \tag{9}$$

This gives $$x=a+b\sqrt{-\ln(R_1)} \tag{10}$$

Steps for the Selection-Rejection Methodology are as follows

Step (1):-Let X be a continuous random variable with probability distribution function given by

$$f(x)=\frac{2(x-a)}{b}\exp\left[-\left(\frac{x-a}{b}\right)^2\right] \quad for \quad x\geq a$$

Step (2):- Let Y be another continuous random variable with probability distribution function given by

$$g(x)=\frac{x-a}{b}$$

Step (3):- Let $0<R_1<1$ be a random number.

Step (4):- Set $X=a+b\sqrt{-\ln(R_1)}$.

Step (5):- If $R_1\leq\frac{f(Y)}{kg(Y)}$,then set $X=Y$ and select the random variable Y;otherwise reject Y and repeat the process from step(1).

V.APPLICATION TO EXPONENTIAL DISTRIBUTION.

Exponential distribution is given by

$$f\left(x\right)=\frac{1}{b}\exp\left[-\left(\frac{x-a}{b}\right)\right] \quad for \quad x\geq a \tag{11}$$

Let $g(x)=\frac{1}{b}$ be the proposal function. (12)

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It may be noted here that $f\left(x\right)$ is the target function.

Then $$h\left(x\right)=\frac{f\left(x\right)}{g\left(x\right)}=\exp\left[-\left(\frac{x-a}{b}\right)\right] \tag{13}$$

$h(x)$ attains its maximum value at $x=a$.

Maximum value of $h(x)$ is 1

Taking , $k=1$, we get

$$\frac{f(x)}{kg(x)}=\exp\left[-\left(\frac{x-a}{b}\right)\right].$$

Let $0<R_1<1$ be a random number

Let $$\exp\left[-\left(\frac{x-a}{b}\right)\right]=R_1 \tag{14}$$

This gives $$x=a-b\ln(R_1) \tag{15}$$

Steps for the Selection-Rejection Methodology are as follows

Step (1):-Let X be a continuous random variable with probability distribution function given by

$$f(x)=\frac{1}{b}\exp\left[-\left(\frac{x-a}{b}\right)\right] \quad for \quad x\geq a$$

Step (2):- Let Y be another continuous random variable with probability distribution function given by

$$g(x)=\frac{1}{b}$$

Step (3):- Let $0<R_1<1$ be a random number.

Step (4):- Set $X=a-b\ln(R_1)$.

Step (5):- If $R_1\leq\frac{f\left(Y\right)}{kg\left(Y\right)}$,then set $X=Y$ and select the random variable Y ;otherwise reject Y and repeat the process from step(1).

VI. APPLICATION TO CAUCHY DISTRIBUTION.

Cauchy distribution is given by

$$f\left(x\right)=\frac{1}{\pi b}\left[1+\left(\frac{x-a}{b}\right)^2\right]^{-1} \qquad for \quad -\infty<x<\infty \tag{16}$$

Let $$g\left(x\right)=\frac{1}{b} \text{ be the proposal function.} \tag{17}$$

It may be noted here that $f\left(x\right)$ is the target function.

Then $$h(x)=\frac{f(x)}{g(x)}=\frac{1}{\pi}\left[1+\left(\frac{x-a}{b}\right)^2\right]^{-1} \tag{18}$$

$h(x)$ attains its maximum value at $x=a$.

Maximum value of $h(x)$ is $\frac{1}{\pi}$.

Taking $k = \frac{1}{\pi}$, we get

$$\frac{f(x)}{kg(x)} = \left[1 + \left(\frac{x-a}{b}\right)^2\right]^{-1} \tag{19}$$

Let $0 < R_1 < 1$ be a random number.

$$\text{Let} \quad \left[1 + \left(\frac{x-a}{b}\right)^2\right]^{-1} = R_1 \tag{20}$$

$$\text{This gives} \quad x = a + b\sqrt{\frac{1-R_1}{R_1}} \tag{21}$$

Steps for the Selection-Rejection Methodology are as follows

Step (1):-Let X be a continuous random variable with probability distribution function given by

$$f(x) = \frac{1}{\pi b}\left[1 + \left(\frac{x-a}{b}\right)^2\right]^{-1} \qquad for \quad -\infty < x < \infty$$

Step (2):- Let Y be another continuous random variable with probability distribution function given by

$$g(x) = \frac{1}{b}$$

Step (3):- Let $0 < R_1 < 1$ be a random number.

Step (4):- Set $X = a + b\sqrt{\frac{1-R_1}{R_1}}$.

Step (5):- If $R_1 \le \frac{f(Y)}{kg(Y)}$,then set $X = Y$ and select the random variable Y ;otherwise reject Y and repeat the process from step(1).

VII. APPLICATION TO BETA DISTRIBUTION.

The probability density function for Beta(α, β) distribution is given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad for \quad 0 < x < 1 \tag{22}$$

Where $\alpha > 0 \;\; and \;\; \beta > 0$

Let $g(x) = 1$ be the proposal density function. (23)

It may be noted here that $f(x)$ is the target function.

Now $h(x) = \frac{f(x)}{g(x)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad for \quad 0 < x < 1$

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With the help of Differential Calculus it can be shown that $h(x)$ attains its maximum value at $x=\frac{\alpha-1}{\alpha+\beta-2}$ and the maximum value of $h(x)$ is given by

$$\frac{\lceil\alpha+\beta}{\lceil\alpha\lceil\beta}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}$$

Let $K=\frac{\lceil\alpha+\beta}{\lceil\alpha\lceil\beta}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}$

Which gives $K=\frac{\lceil\alpha+\beta}{\lceil\alpha\lceil\beta}K'$

$$Where \quad K'=\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}$$

Hence we get $\frac{f(x)}{Kg(x)}=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{K'}$

Steps for the Selection-Rejection Methodology for the Beta(α,β) distribution are as follows.

Step (1):- Let X be a continuous random variable with probability distribution function

$$f(x)=\frac{\lceil\alpha+\beta}{\lceil\alpha\lceil\beta}x^{\alpha-1}(1-x)^{\beta-1} \quad for \quad 0<x<1$$

Where $\alpha>0 \;\; and \;\; \beta>0$

Step (2):- Let Y be another continuous random variable with probability distribution function given by $g(x)=1$.

Step (3):- Let $K=\frac{\lceil\alpha+\beta}{\lceil\alpha\lceil\beta}K'$.

$$Where \; K'=\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}$$

Step (4):- Let $0<r<1$ be a random number.

Step(5):- If $r\le\frac{f(x)}{Kg(x)}\Leftrightarrow r\le\frac{x^{\alpha-1}(1-x)^{\beta-1}}{K'}$,then set $X=Y$ and select the random variable Y ;otherwise reject the random variable Y and repeat the process from Step(1).

VIII. CONCLUSION.

Selection-Rejection Methodology is valid for any dimension of random variable(continuous or discrete).In this method we approximate the target function to proposal function so that after a number of successive iterations the proposal function becomes almost equal to target function and proposal function is selected.

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