Thermal Deflection of a Thin Clamped Hollow Circular Disk due to Heat Generation by Quasi-static Approach

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Abstract— This paper deals with the determination of the thermal deflection in a thin clamped hollow circular disk defined as $a \le r \le b$; $0 \le z \le h$ under unsteady temperature field due to internal heat generation within it. A thin hollow circular disk is considered having arbitrary initial temperature and subjected to heat flux at the outer circular boundary $(r = b)$ where as inner circular boundary $(r = a)$ is at zero heat flux. Also, the upper surface $(z = h)$ of the hollow circular disk is at zero temperature and the lower surface $(z = 0)$ is insulated. The governing heat conduction equation has been solved by using integral transform technique. The inner and outer edges of the circular disk are clamped $\frac{\partial \omega}{\partial r} = 0$ at $r = a,\ r = b$. The results are obtained in series form in terms of Bessel's functions and are illustrated graphically.

Keywords- *Transient, Thermoelastic problem, Thermal Stresses, Heat generation, Hollow circular disk*

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I. INTRODUCTION

Boley and Weiner [1] studied the problems of thermal deflection of an axisymmetric heated circular plate in the case of fixed and simply supported edges. Roy choudhury [2] discussed the normal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. This satisfies the time-dependent heat conduction equaton. Deshmukh and Khobragade [3] has determined a quasi-static thermal deflection in a thin circular plate due to partially distributed and axisymmetric heat supply on the outer curved surface with the upper and lower faces at zero temperature. Deshmukh et al. [5] has determined the thermal stresses in a hollow circular disk due to internal heat generation within it. Recently Deshmukh et al. [6] studied the thermal deflection in a thin circular plate subjected to heat generation within In this paper the work of Deshmukh et al. [6] has been extended for two dimensional non-homogeneous boundary value problem of heat conduction and studied the thermal deflection of thin clamped hollow circular disk defined as $a \le r \le b$; $0 \le z \le h$ due to internal heat generation within it. A thin hollow circular disk is considered having arbitrary initial temperature and subjected to heat flux at the outer circular boundary $(r = b)$ where as inner circular boundary $(r = a)$ is at zero heat flux. Also, the upper surface $(z = h)$ of the hollow circular disk is at zero temperature and the lower surface $(z = 0)$ is insulated. The governing heat conduction equation has been solved by using integral transform technique. The results are obtained in series form in terms of Bessel's functions. The results for thermal deflection have been computed numerically and are illustrated graphically. It is believe that this particular problem has not been previously considered The rotating hollow circular disk is having the applications in Aerospace engineering particularly in gas turbines and gears. The hollow circular disk is normally work under thermo-mechanical loads .

II. FORMULATION OF THE PROBLEM

A Heat Conduction Equation

Consider a thin hollow circular disk of thickness h occupying space D defined by $a \le r \le b$; $0 \le z \le h$ Initially, the disk is kept at arbitrary temperature $F(r, z)$. The inner circular boundary $(r = a)$ is at zero temperature whereas the heat flux $\frac{Q(z,t)}{k}$ is applied on the outer circular boundary $(r = b)$. Also, the upper surface $(z = h)$ of the hollow circular disk is at zero temperature and the lower surface $(z = 0)$ is insulated. For time $t > 0$, heat is generated within the thin hollow circular disk at the rate $g(r, z, t)$. Under these conditions, the thermal deflection in a thin hollow circular disk due to heat generation is required to be determined.

The temperature of the hollow circular disk $T(r, z, t)$ at time t satisfying the differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r,z,t)}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

with the boundary conditions,

$$k\frac{\partial T}{\partial r} = 0 \quad \text{at } r = a, \quad t > 0 \tag{2}$$

$$k\frac{\partial T}{\partial r} = Q(z,t) \quad \text{at } r = b, \quad t > 0 \tag{3}$$

$$\frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0, \quad t > 0 \tag{4}$$

ISSN: 2321-8169
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$$T = 0 \quad \text{at} \quad z = h, \quad t > 0 \tag{5}$$

and initial condition

$$T(r,z,t) = F(r,z), \text{in } a \le r \le b,\ 0 \le z \le h \quad \text{for} \quad t = 0 \tag{6}$$

where K and α are thermal conductivity and thermal diffusivity of the material of the hollow circular disk respectively.

B Thermal Deflection

The differential equation satisfying the deflection function $\omega(r,t)$ is given as

$$\nabla^4 \omega = -\frac{\nabla^2 M_T}{D(1-\nu)} \tag{7}$$

where, M_T is the thermal moment of the disk defined as

$$M_T = a_t E \int_0^h T(r,z,t)\, z\, dz \tag{8}$$

D is the flexural rigidity of the disk denoted as

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{9}$$

a_t, E and ν are the coefficients of the linear thermal expansion, the Young's modulus and Poisson's ratio of the disk material respectively and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{10}$$

Since, the inner and outer edges of the hollow circular disk are clamped;

$$\frac{\partial \omega}{\partial r} = 0 \quad \text{at} \quad r = a \ \text{and}\ r = b \tag{11}$$

Initially $T = \omega = F(r,z)$ at $t = 0$

Equations (1) to (11) constitute the mathematical formulation of the thermoelastic problem in a thin hollow circular disk.

III .SOLUTION

A. Heat Conduction Equation

To obtain the expression for temperature function $T(r,z,t)$, we develop the finite Fourier transform and finite Hankel transform and their respective inverses and operate them on the Eqs. (1-6), one obtains the expression of temperature distribution as

$$\begin{aligned} T(r,z,t) = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} K(\eta_p,z) K_0(\beta_m,r) e^{-\alpha(\beta_m^2+\eta_p^2)\,t} \\ \times \left\{ \int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m,r') K(\eta_p,z') F(r',z')\, dr'\, dz' \right. \\ + \int_{t'=0}^{t} e^{\alpha(\beta_m^2+\eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m,r') K(\eta_p,z') g(r',z',t')\, dr'\, dz' \right. \\ \left.\left. + \frac{\alpha}{K} b K_0(\beta_m,b) \int_{z'=0}^{h} K(\eta_p,z') Q(z',t')\, dz' \right) dt' \right\} \end{aligned} \tag{12}$$

where $$K(\eta_p,z) = \sqrt{\frac{2}{h}} \cos(\eta_p z) \tag{13}$$

and $\eta_1, \eta_2, \ldots\ldots$ are the positive roots of the transcendental equation

$$\cos(\eta_p h) = 0, \quad p = 1,2,\ldots\ldots \tag{14}$$

$$K_0(\beta_m,r) = \frac{\pi}{\sqrt{2}} \frac{\beta_m . J_0'(\beta_m b) Y_0'(\beta_m b)}{\left[1 - \frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{\frac{1}{2}}} \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right] \tag{15}$$

and $\beta_1, \beta_2, \ldots\ldots$ are the positive roots of the transcendental equation

$$\frac{J_0'(\beta a)}{J_0'(\beta b)} - \frac{Y_0'(\beta a)}{Y_0'(\beta b)} = 0 \tag{16}$$

B. Thermal Deflection

Using Eq. 16 into Eq. 8, one obtains

$$\begin{aligned} M_T = \sqrt{\frac{2}{h}} a_t E h \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin(\eta_p h)}{\eta_p} K_0(\beta_m,r) e^{-\alpha(\beta_m^2+\eta_p^2)\,t} \\ \times \left\{ \int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m,r') K(\eta_p,z') F(r',z')\, dr'\, dz' \right. \\ + \int_{t'=0}^{t} e^{\alpha(\beta_m^2+\eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m,r') K(\eta_p,z') g(r',z',t')\, dr'\, dz' \right. \\ \left.\left. + \frac{\alpha}{K} b K_0(\beta_m,b) \int_{z'=0}^{h} K(\eta_p,z') Q(z',t')\, dz' \right) dt' \right\} \end{aligned} \tag{17}$$

Assume the solution of Eq.7 satisfy condition 11 as

$$\omega(r,t) = \sum_{m=1}^{\infty} C_m(t) \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right] \tag{18}$$

where β_m' are the positive roots of the transcendental equation

$$\frac{J_0'(\beta a)}{J_0'(\beta b)} - \frac{Y_0'(\beta a)}{Y_0'(\beta b)} = 0.$$

It can be easily shown that

$$\frac{\partial \omega}{\partial r} = \sum_{m=1}^{\infty} C_m(t) \left[\frac{J_0'(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0'(\beta_m r)}{Y_0'(\beta_m b)} \right]$$

$$\frac{\partial \omega}{\partial r} = 0 \text{ at } r = a$$

Now

$$\frac{\partial \omega}{\partial r} = \sum_{m=1}^{\infty} C_m(t) \left[\frac{J_0'(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0'(\beta_m r)}{Y_0'(\beta_m b)} \right]$$

$$\frac{\partial \omega}{\partial r} = 0 \text{ at } r = b.$$

Hence the solution 18 satisfies the condition 11.
Now,

$$\nabla^4 \omega = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 \sum_{m=1}^{\infty} C_m(t) \times \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right] \qquad (19)$$

Using the well known result

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) J_0(\beta_m r) = -\beta_m^2 J_0(\beta_m r)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) Y_0(\beta_m r) = -\beta_m^2 Y_0(\beta_m r)$$

in Eq.19, one obtains

$$\nabla^4 \omega = \sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right]$$

Also

$$\nabla^2 M_T = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \sqrt{\frac{2}{h}}\, a_t\, E h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\eta_p} \sin(\eta_p h) \times K_0(\beta_m, r) e^{-\alpha(\beta_m^2 + \eta_p^2)t}$$
$$\times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') F(r', z') dr' dz' + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' + \frac{\alpha}{K} b K_0(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z') Q(z', t') dz' \right) dt' \right\}$$

$$\nabla^2 M_T = -\sqrt{\frac{2}{h}}\, a_t\, E h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\eta_p} \sin(\eta_p h) \beta_m^2 K_0(\beta_m, r) e^{-\alpha(\beta_m^2 + \eta_p^2)t}$$
$$\times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') F(r', z') dr' dz' + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' + \frac{\alpha}{K} b K_0(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z') Q(z', t') dz' \right) dt' \right\} \qquad (20)$$

Substituting Eq.19 and Eq.20 into Eq.7, one obtains

$$\sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right]$$
$$= \sqrt{\frac{2}{h}} \frac{a_t E}{D(1-v)} h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\eta_p} \sin(\eta_p h) \beta_m^2 \frac{\pi}{\sqrt{2}} \times \frac{\beta_m . J_0'(\beta_m b) . Y_0'(\beta_m b)}{\left[1 - \frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)} \right]^{\frac{1}{2}}} \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right]$$
$$\times e^{-\alpha(\beta_m^2 + \eta_p^2)t} \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') F(r', z') dr' dz' + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' + \frac{\alpha}{K} b K_0(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z') Q(z', t') dz' \right) dt' \right\} \qquad (21)$$

Solving equation (21), one obtain

$$C_m(t) = \sqrt{\frac{2}{h}} \frac{a_t Eh}{D(1-v)} \sum_{p=1}^{\infty} \frac{1}{\eta_p} \cos(\eta_p h) \frac{1}{\beta_m^2} \frac{\pi}{\sqrt{2}} \times \frac{\beta_m . J_0'(\beta_m b) . Y_0'(\beta_m b)}{\left[1 - \frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)} \right]^{\frac{1}{2}}} e^{-\alpha(\beta_m^2 + \eta_p^2)t}$$
$$\times \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') F(r', z') dr' dz' + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' + \frac{\alpha}{K} b K_0(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z') Q(z', t') dz' \right) dt' \right\} \qquad (22)$$

Substituting equation (22) into equation (18), one obtains

$$\omega(r,t) = \sqrt{\frac{2}{h}} \frac{a_t Eh}{D(1-v)} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\eta_p} \cos(\eta_p h) \frac{1}{\beta_m .} \frac{\pi}{\sqrt{2}} \times \frac{J_0'(\beta_m b) . Y_0'(\beta_m b)}{\left[1 - \frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)} \right]^{\frac{1}{2}}} \times \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right]$$
$$\times \left[e^{-\alpha(\beta_m^2 + \eta_p^2)t} \left\{ \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') F(r', z') dr' dz' + \int_{t'=0}^{t} e^{\alpha(\beta_m^2 + \eta_p^2)t'} \left(\frac{\alpha}{K} \int_{r'=a}^{b} \int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' + \frac{\alpha}{K} b K_0(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z') Q(z', t') dz' \right) dt' \right\} \right. \qquad (23)$$

IV. SPECIAL CASE

Setting:

- $F(r,z)=\left(r^2-a^2\right)^2\times r^2\times z^2\times(z-h)^2$
- $Q(z,t)=z^2\times(z-h)^2\times e^{-\phi t}$
- $g(r,z,t)=g_{pi}.\delta(r-r_1).\delta(z-z_1).\delta(t-\tau)$
- where r is the radius measured in meter, δ is the Dirac-delta function, $\phi=5\ \ s^{-1}>0$.
- The heat source $g(r,z,t)$ is an instantaneous point heat source of strength $g_{pi}=50\ \ J\,m^{-1}$ situated at the centre of the hollow circular disk along the radial direction and axial direction and releases its heat spontaneously at time $t\to\tau=5\sec$.
- Dimension
- Inner radius of a thin hollow circular disk $a=1\ \ m$
- Outer radius of a thin hollow circular disk $b=2\ \ m$
- Thickness of a thin hollow circular disk $z=0.4\ m$
- Central circular path of disk in radial and axial directions $r_1=1.5\ \ m$ and $z_1=0.2\ \ m$
- Material properties
- The numerical calculation has been carried out for a copper (pure) thin hollow circular disk with the material properties
- Thermal diffusivity $\alpha=112.34\times10^{-6}\ \ \ m^2s^{-1}$
- Thermal conductivity $k=386\ \ \ W\,m^{-1}K^{-1}$
- Density $\rho=8954\ \ \ kg\,m^{-3}$
- Specific heat $c_p=383\ \ \ J\,kg^{-1}K^{-1}$
- Poisson ratio $\nu=0.35$
- Coefficient of linear thermal expansion, $a_t=16.5\times10^{-6}\ \ K^{-1}$
- Lame' constant $\mu=26.67$.
- Roots of the transcendental equation
  Let
- β_1=3.1965 , β_2=6.3123 , β_3=9.4445, β_4=12.5812 , β_5=15.7199

  are the positive roots of transcendental equation
- $$\left(\frac{J_0'(\beta_m)}{J_0'(2\beta_m)}-\frac{Y_0'(\beta_m)}{Y_0'(2\beta_m)}\right)=0$$
- The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2000 and the graphs are plotted with the help of Excel (MS office-2000).
- For convenience setting $X=\dfrac{a_t\,E\,h}{D(1-\nu)}$ i.e. elastic material constants.

V. CONCLUDING REMARKS

In this paper we extend the work of Deshmukh et.al [6] in two dimensional inhomogeneous boundary value problem of heat conduction in a thin hollow circular disk and determined the expressions for temperature, deflection due to internal heat generation within it.

As a special case, a mathematical model is constructed for Copper (Pure), thin clamped hollow circular disk with the material properties specified as above. The heat source is an instantaneous point heat source of strength g_{pi} situated at the center of the circular disk along radial and axial direction and releases its heat instantaneously at the time $t\to\tau$.

From figure 1, It is observed that there is a variation in temperature within the regions given as $1\le r\le1.2,\ 1.2\le r\le1.6,\ 1.6\le r\le1.9$ due to heat generation. Also, observed that the heat flows towards outer circular edge of hollow circular disk, clamped support will not allow and hence heat flows moves towards lower surface in axial directions.

From figure 2, It is observed that the deflection is maximum at the inner circular edge and decreases from inner to outer circular edge.

We can summaries that due to internal heat generation within the thin hollow circular disk, the deflection occurs at inner circular edge. The direction of heat flow and direction of body deflection are same and they are proportionate.

The rotating hollow circular disk is having the applications in Aerospace engineering particularly in gas turbines and gears. The hollow circular disk is normally work under thermo-mechanical loads.

Also any particular case of special interest can be derived by assigning suitable values to the parameter and function in the expressions 12 and 23

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ISSN: 2321-8169
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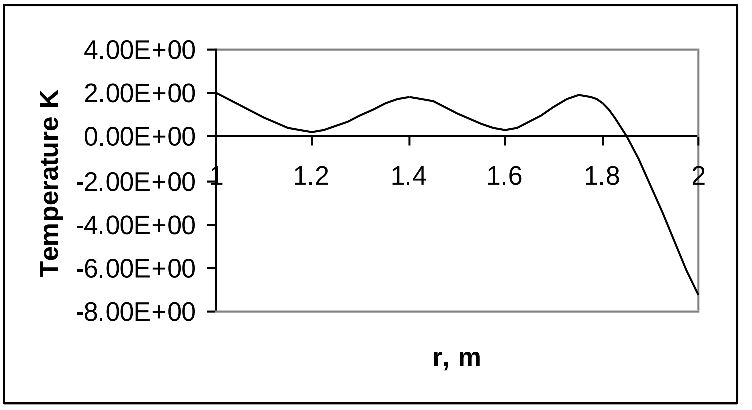

Fig. 1 Temperature distribution for various values of r

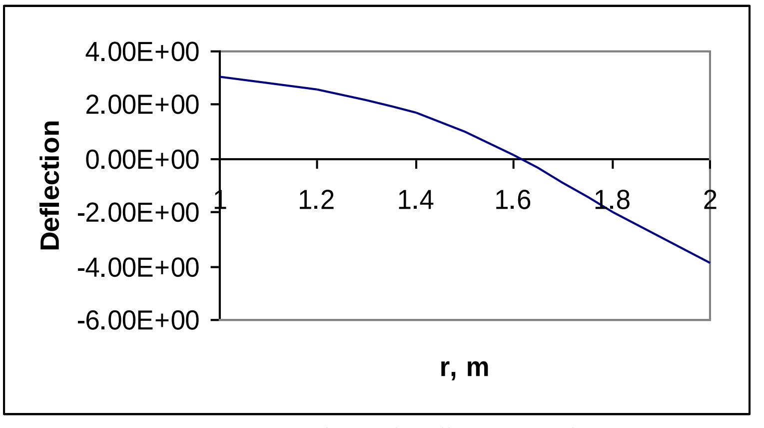

Fig. 2 Thermal deflection for different values of r.