Novel Approaches in Optimal Control and Their Application to Find Analytical Solutions for Minimum-Time Ascending Maneuver

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Abstract**—**Of utmost importance to development of UAVs, is an automatic flight system. Minimum-time maneuvers however, are most challenging due to the fact that dynamic and kinematic constraints should not be violated while conservative assumptions on actuator limits, compromise optimality. A major hindrance in minimum time to climb is that the kinematic constraint introduces a redundant state which makes the Hamiltonian quite difficult to handle while numerical methods are distrustful for that the resulting system of two-point boundary value problem is unbounded. To overcome these, a novel approach to a class of problems will be suggested. It is proved herein how this method allows the cost functional to be changed so that the number of state variables is reduced. These results facilitate the finding of the analytical optimal solution to reaching a waypoint with any initial but no final boundary conditions on the angle of trajectory. For when the final angle of the trajectory is specified, a method will be proposed. The solutions found here can be used by an RHPC method.

Keywords- *aircraft, automatic flight, minimum time, optimal control, time optimal, UAV*

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I. Introduction

The exceedingly important problem of path planning can be a demanding job for UAVs. Existence of an analytical solution makes the implementation of an RHPC method more practical. Depending on the task of a UAV and its environment, an ascending maneuver to reach a waypoint in minimum-time becomes necessary. From encountering an obstacle in a time-critical circumstance to tasks which UAVs can only perform while not being in a maneuver to the problem of changing altitude in minimum time, the need for the said scenario rears. Therefore, a minimum-time to “climb” has interested researchers in the field as long ago as the seventies [1-2]. The singular perturbation method reduces the order of the problem by only considering the fast states. In addition, reduction of the number of state variables by combining the inertial “height” and velocity into one energy variable has gained some popularity [3-4]. An approach with Hamiltonian has also been considered a number of years ago where the conclusion was a bang-bang control [5]. But as is stated herein, bang-bang control is only the solution when the maximum thrust exceeds maximum lift at all times. Analytical solution of this problem has been avoided for the mathematical complexities that arise. Herein, an approach for such time-optimal problems with one “redundant” state is put forth. In the case where the final angle of trajectory is free, this method allows for searching for the time-optimal control, in a family of optimal controls of another simpler problem. Reformulation of the problem rewarded by this method is then dealt with analytically so that the family of time-optimal controls (for different waypoints) is found. For the case where the final angle of the trajectory is specified, a locally optimal method will be proposed that yields the global optimum if one conjecture is true. For more tangibility, the mathematical method here is expressed in the context of this problem.

II. Problem Formulation with Dynamic and Kinematic Constraints

The inertial axis system used here has its origin coinciding with aircraft’s center of mass at $t=0$, the $x-axis$ is in the plane of symmetry of the aircraft, parallel to flatland. The $z-axis$ is also in the plane of symmetry of the aircraft heading the sky. The $y-axis$ completes the right-handed system. Since the ascending maneuver is in the $x-z$ plane, the angle between the velocity vector and inertial $x-axis$ will be denoted by θ and the final time by t_f .

The goal is then to find a control that minimizes $J=\int_0^{t_f} dt$ subject to aerodynamic and kinematic constraints as well as $z(t_f)=c$ and $\theta(t_f)=0$. $x(t_f)$ is found as a byproduct and can be used if there actually is an obstacle so as to signal the point for starting the maneuver such that $x(t_f)$ is as specified. Analytical time-optimal solution with $\theta(t_f)$ free will be found as well.

The forces acting on an aircraft are the gravitational force, thrust and aerodynamic force. The latter two can be controlled, though with dynamic constraints. Thrust (denoted by T) acts in the direction of the relative wind and herein is assumed to belong in the admissible set $0 \le T \le T_{\max}$. Aerodynamic force is decomposed into lift (denoted by L) which acts perpendicular to the relative wind and drag (denoted by D) which resists movement [6-7].

Lift and drag are respectively given by:

$$L=\frac{1}{2}\rho S v^2 C_l \tag{1}$$

$$D=\frac{1}{2}\rho S v^2 C_D \tag{2}$$

where ρ , S , v , C_l and C_D respectively denote air density, wing area, true speed, lift coefficient and drag coefficient. The latter two depend on Reynolds number (Re), Mach number (M) and angle of attack (α). Since both the drag and lift coefficients depend on the angle of attack, the drag coefficient can be expressed as a function of lift coefficient. Drag polar does just that and is given by:

$$C_D=C_{D_{\min}}+\Delta C_{D_{\min}}+K(C_l-C_{l_0})^2 \tag{3}$$

where K is a dimensionless parameter, depending on aircraft wing dimensions, Reynolds number, Mach number and the aircraft configuration. C_{l_0} is the lift coefficient at zero angle

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of attack, $C_{D_{\min}}$ is the parasitic drag coefficient and the term $\Delta C_{D_{\min}}$ is added to compensate for cambered wing [8]. Although C_{D_0} is frequently used for zero-lift drag coefficient, it will be used in this text as the sum of the last two terms, thus substituting (3) into (2) gives:

$$D=\frac{1}{2}\rho S v^2(C_{D_0}+K(C_l-C_{l_0})^2) \qquad (4)$$

Fig. 1 shows an instance in climb. Equations of motion are given by:

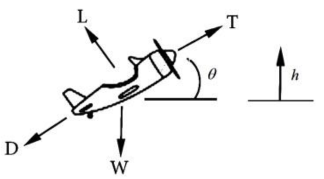

Figure 1. An Instance of Climb

$$L-W\cos\theta=mv\dot{\theta} \qquad (5)$$
$$T-D-W\sin\theta=m\dot{v} \qquad (6)$$

Considering the kinematic equation $\dot{z}=v\sin\theta$, by substituting (1) and (4) in (5) and (6), the state equations are:

$$\begin{bmatrix}\dot{z}\\ \dot{v}\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix} v\sin\theta \\ \frac{1}{m}(u_1-\frac{1}{2}\rho S v^2(C_{D_0}+K(u_2-C_{l0})^2)-W\sin\theta) \\ \frac{1}{mv}(\frac{1}{2}\rho S v^2 u_2 - W\cos\theta)\end{bmatrix} \qquad (7)$$

where $\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}T\\ C_l\end{bmatrix}$.

The problem is then to find controls u_1, u_2 so as to:

$$\text{Minimize } J=\int_0^{t_f} dt \text{ subject to:}$$

$$0\le u_1\le T_{\max}$$
$$c_{\min}\le u_2\le c_{\max} \qquad (8)$$

$$x(0)=y(0)=z(0)=0$$
$$v(0)=v_0$$
$$z(t_f)=c \qquad (9)$$
$$\theta(0)=\theta_0$$

$$\theta(t_f)=0 \ \ and\ \ \theta(t_f)\ \ free$$

The last line of (9) indicates that two problems will be handled here. The assumptions for solving the problem are as follows:

K and C_{D_0} are treated as constant. This is true approximately up to 0.75 Mach [8-9] after which, these parameters increase. Although many UAVs fly well under this speed, the methods and solutions found here can be extended to higher speeds.

Mass, air density and the gravitational field are assumed to be constants.

III. Solution

The analytical solution for when $\theta(t_f)$ is free will be found first, but not before some mathematical methods are proposed.

A. *Laying the Mathematical Foundations*

Here a method is presented so as to eliminate the redundant state. Suppose the goal is to:

$$\text{Minimize } J=\int_0^{t_f} dt \text{ subject to:}$$

$$\underline{\dot{x}}_{aug}=\begin{bmatrix}\dot{x}_1\\ \underline{\dot{x}}\end{bmatrix}=a'(\underline{x},u,t) \qquad (10\text{-a})$$

$$\underline{x}_{aug}(0)=\begin{bmatrix}0\\ \underline{x}_i\end{bmatrix},\ \ \underline{x}_{aug}(t_f)=\begin{bmatrix}c\\ free\end{bmatrix} \qquad (10\text{-b})$$

where a' is a general non-linear function, x_{aug} denotes all of the states and x_1 is a state whose value is known at both the initial and final times. Note also that x_1 is the only state with a specified final-time boundary value and can be viewed as a redundant state since its value does not affect other states (see the argument of a'). It is assumed for convenience that $x_1(0)=0$, $x_1(t_f)=c$, $c\succ 0$.

By writing the first row of (9) as $\dot{x}_1=f(\underline{x},u,t)$, this problem can be formulated with an integral boundary value constraint, namely to minimize $J=\int_0^{t_f} dt$, subject to:

$$\underline{\dot{x}}=a(\underline{x},u,t)$$

$$\int_0^{t_f} f(\underline{x},u,t)dt=c,\ \ \underline{x}_{aug}(0)=\begin{bmatrix}0\\ x_i\end{bmatrix} \qquad (11)$$

where now the state variables are free at the final time. If u_1^* is an optimal control with optimal time t^*, then: $c=\int_0^{t^*} f(\underline{x},u_1^*,t)dt$. Argument $\underline{x}$ will be dropped from this point forth for convenience and some needed notations will follow that are chosen to correspond with the problem at hand here:

Def. 1: For $f(u,t)$, $v(u,t)$ is defined as $v(u,t)=\int_0^t f(u,\tau)d\tau$ (velocity function).

Def. 2: U denotes the set of all admissible controls and unless stated otherwise, is taken to be all the functions that are piece-wise continuous on $(0,\infty)$ whose jumps are bounded, i.e.:

$\forall\tau\succ 0:\left|\lim_{t\to\tau^-} f(u,t)-\lim_{t\to\tau^+} f(u,t)\right|\prec\infty$. Therefore $v(u,t)$ is continuous on $(0,\infty)$.

Def. 3: $u_v^*(t_{f,a},t)$ denotes a control that is defined on $(0,\infty)$ such that:

$\forall u\in U: v(u_v^*(t_{f,a},t),t_{f,a})\ge v(u(t),t_{f,a})$. In other words $u_v^*(t_{f,a},t)$ maximizes v at $t=t_{f,a}$. If $c\prec 0$, a similar definition requires minimizing v.

Def. 4: $U'(t')$ is a set consisting of $u_v^*(t_{f,a},t)$ for $0\le t_{f,a}\le t'$ in which there is one-to-one correspondence between $u_v^*(t_{f,a},t)$ and $t_{f,a}$, such that on the ball $t_{f,a}\in(0,t')$ and $t\in(0,t')$, the functional $v(u_v^*(t_{f,a},t),t)$ is continuous.

Lemma 1: A minimum-time solution to (10-a) and (10-b) is $u_v^*(t^*)$ where t^* is the optimal time.

Proof: By definition, $v(u_v^*(t^*),t^*)\ge c$. If however $v(u_v^*(t^*),t^*)\succ c$ then since $v(u_v^*(t_{f,a},t),t)$ is continuous, there exists $t_2^*\prec t^*$ such that $v(u_v^*(t^*),t_2^*)=c$ which contradicts the optimality of t^*. ■

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Therefore, another approach to finding a time-optimal solution to (10-a) and (10-b) is as follows.

Consider $v(u,t_f)=\int_0^{t_f} f(u,t)dt$ subject to the constraints (11) and the set of admissible controls U is as in Def. 2. With an integral constraint at final time, minimizing $J=\int_0^{t_f} dt$ with free final time can be rewritten as maximizing $v=\int_0^{t_f} \mathrm{f(u,t)dt}$ for different fixed final times. The shortest final time $t_{f,a}$ for which the maximizing control, gives the penalty $v(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}),\mathrm{t}_{f,a})=\mathrm{c}$ is the time-optimal control. This assumed final time is also the shortest time among all the times for which the controls of the set U' satisfy the boundary value. In other words: $t^*=\inf_{t_{f,a}}\{t_f : v(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}),t_f)=c\}$.

This way, the number of state variables is reduced by one. The physical concept behind this as an example is that if the time-optimal control for a car to reach some speed v_f on the positive x-axis, takes the time t^*, then the maximizing control for the different cost function "velocity" at the fixed final time t^*, has the penalty v_f.

However, it is the "distance" not the "velocity" which is specified at the final time. Nevertheless, a theorem follows that expands this method to "distance." Conditions of the theorem are sufficient to say that the time-optimal control belongs in the set of controls that maximize "velocity" for an assumed final time shorter than t^*. Thus, the "distance" functional is defined next.

Def. 5: The functional $z(\mathrm{u},\mathrm{t}_f)$ is defined as $z(u,t_f)=\int_0^{t_f} v(u,t)dt$ and the control that satisfies $t^*=\inf_u\{\mathrm{t}_f : z(u,\mathrm{t}_f)=c\succ 0\}$ is denoted by u_z^*. In other words u_z^* is the minimum-time control that satisfies the boundary value condition $z(\mathrm{u},\mathrm{t}_f)=\mathrm{c}$.

Lemma 2: Suppose that U' is chosen such that $f(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)\succ 0$ and $\dot f(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)\le 0$. If $v(u_z^*,t)$ is concave, and $u_z^*\notin U'(t^*)$, then for all $t_{f,a}$, there exists one and only one positive t_c such that $v(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t_c)=v(u_z^*,t_c)$.

Proof: Since $f(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)\succ 0$ and $\dot f(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)\le 0$, then $v(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)$ is strictly increasing and concave. So is $v(u_z^*,t)$. Since for all $t_{f,a}$, at $t=t_{f,a}$, $v(u_v^*(\mathrm{t}_{f,a}),t)\ge v(u_z^*,t)$ holds, if there is no such t_c then the inequality holds for all t, that is $\int_0^{t^*} v(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)dt\ge\int_0^{t^*} v(u_z^*,t)dt$ which in case of equality contradicts $u_z^*\notin U'(t^*)$ and otherwise contradicts the optimality of u_z^*. Furthermore, since $v(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)$ is increasing and concave, then there cannot be more than one t_c for each $u_v^*(t_{f,a},\mathrm{t})$.

Theorem 1: Suppose that U' is chosen such that $f(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)\succ 0$ and $\dot f(u_v^*(\mathrm{t}_{f,a},\mathrm{t}),t)\le 0$. If $v(u_z^*,t)$ is concave, then there exists $t_{f,a}$ such that $t_{f,a}\le t^*$ and $z(u_v^*(t_{f,a},\mathrm{t}),t^*)=z(u_z^*,t^*)$.

Proof:

It is assumed that $u_z^*\notin U'(t^*)$ (Otherwise existence is achieved). For clarity the point where $v(u,t)=v(u_z^*,t)$ will be denoted by $t_c(\mathrm{u})$. Lemma 2 assures the existence of such point for all members of $U'(\mathrm{t}^*)$. Now consider the functionals as defined below:

$$\Delta J(u)=\int_0^{t_c(\mathrm{u})} v(u,t)dt-\int_0^{t_c(\mathrm{u})} v(u_z^*)dt$$

$$\Delta J'(u)=\int_{t_c(\mathrm{u})}^{t^*} v(u,t)dt-\int_{t_c(\mathrm{u})}^{t^*} v(u_z^*)dt$$

Obviously, these two functionals have different signs for any given $u_v^*(\mathrm{t}_{f,a},\mathrm{t})$. Consider $u_v^*(t^*)$ (Fig. 2). Since by definition of u_z^*, $v(u_v^*(t^*),t^*)\ge v(u_z^*,t^*)$, then $t_c(u_v^*(t^*,\mathrm{t}))\le t^*$ (or else $z(u_z^*,t^*)\le z(u_v^*(t^*,\mathrm{t}),t^*)$ which either achieves existence or contradicts optimality of u_z^*).

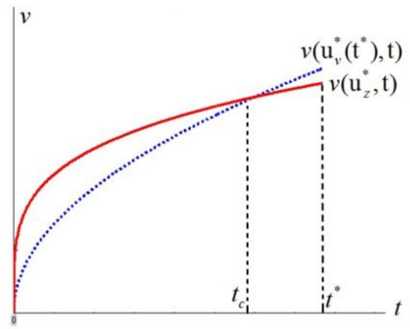

Figure 2. Qualitative Illustration of $v(\mathrm{u}_v^*(\mathrm{t}^*),\mathrm{t})$ and $v(\mathrm{u}_z^*,\mathrm{t})$

Therefore, if $t_{f,a}\ge t^*$, then $\Delta J'(u_v^*(t_{f,a},\mathrm{t}))\ge 0$. Thus all the controls for which $\Delta J\succ 0\Rightarrow\Delta J'\prec 0$ belong to $U'(t^*)$ (Def. 4). Now, in this set $\lim_{t_{f,a}\to 0^+}\Delta J'(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))\prec 0$ and $\lim_{t_{f,a}\to t^{*-}}\Delta J'(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))\succ 0$. Thus the continuous functional $\Delta J'$, changes sign from negative to positive. Hence, there exists $t_{f,a}$ in this set such that $\Delta J'(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))=0$. Also since $\Delta J(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))$ always has a sign opposite to $\Delta J'(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))$, it is further deduced that for this same $t_{f,a}$, $\Delta J(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))=0$. Therefore at this $t_{f,a}$:

$$\Delta J(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))+\Delta J'(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}))=0\Rightarrow\int_0^{t^*} v(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}),\mathrm{t})\mathrm{dt}=\int_0^{t^*} v(\mathrm{u}_z^*,\mathrm{t})\mathrm{dt}$$

$$\Rightarrow \mathrm{z}(\mathrm{u}_v^*(\mathrm{t}_{f,a},\mathrm{t}),\mathrm{t}^*)=\mathrm{z}(\mathrm{u}_z^*,\mathrm{t}^*)$$

The proof is complete now.■

In short, with an integral constraint at final time, minimizing $J=\int_0^{t_f} dt$ with free final time can be rewritten as maximizing $v=\int_0^{t_f}\mathrm{f(u,t)dt}$ for different fixed final times in the set U' (if it exists) so as to eliminate the redundant state. Then the shortest time among all optimal controls for different fixed final times for which the integral constraint is satisfied constitutes to a time-optimal control. Furthermore, if the constraint involves twice integration and the theorem's hypothesis is met, then the solution lies within the set of optimal controls for the corresponding problem with one integration.

B. B. Analytical Solution of (8) and (9) When $\theta(t_f)$ Is Free

In light of the mathematical discussions earlier, the problem of (8) and (9) can be written with an integral constraint. If the resultant force in the direction of the $z-axis$ is denoted by F_z, then:

$$F_z = L\cos\theta + (T-D)\sin\theta - W \quad (12)$$

where $\theta(t)$ denotes the angle between the wind-axis system and the inertial-axis system. Substituting (1) and (4) in (12) we get:

$$F_z = \frac{1}{2}\rho S v^2 u_2\cos\theta + (u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l_0})^2))\sin\theta - W \quad (13)$$

where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} T \\ C_l \end{bmatrix}$. Thus:

$$z = \frac{1}{m}\int_0^{t_f}\int_0^{t_{f1}}(\frac{1}{2}\rho S v^2 u_2\cos\theta + (u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l_0})^2))\sin\theta - W)dtdt_{f1} \quad (14)$$

If t^* is the optimal time and the set U' exists, use of the theorem gives: $t^* = \inf_{t_{f,a}}\{t_f : z(u_v^*(t_{f,a}), t_f) = c\}$. Thus to find the set of all $u_v^*(t_{f,a})$, define v_z as follows:

$$v_z = \frac{1}{m}\int_0^{t_{f1}}(\frac{1}{2}\rho S v^2 u_2\cos\theta + (u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l_0})^2))\sin\theta - W)dt \quad (15)$$

Obviously $z = \int_0^{t_f} v_z dt_{f1}$.

It is assumed that throughout the maneuver, $0 \le \theta \le \frac{\pi}{2}$ (climbing). Now if $L_{\max} \ge T_{\max}$, then the optimal control is not a bang-bang control. (7) implies that θ is increasing and hence if $L_{\max} \ge T_{\max}$, (12) implies that F_z is decreasing. Thus, conditions of theorem 1 are met and the solution lies in the set of optimal controls (for different fixed times) of the following problem:

For different fixed final times $t_{f,a}$:

Maximize v_z subject to:

$$\begin{bmatrix} \dot v \\ \dot\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{m}(u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l0})^2) - W\sin\theta) \\ \frac{1}{mv}(\frac{1}{2}\rho S v^2 u_2 - W\cos\theta) \end{bmatrix} \quad (16)$$

$$z(u, t_f) = c, \quad v(0) = v_0, \quad \theta(0) = \theta_0 \quad (17)$$

$$0 \prec u_1 \prec T_{\max}$$
$$C_{l,\min} \prec u_2 \prec C_{l,\max} \quad (18)$$

where now by application of the method proposed in this paper, one redundant state is eliminated and the cost functional has changed. Casting aside (18), unbounded control is assumed for now. The Hamiltonian is as follows (for necessary calculus of variation see [10]):

$$H = \frac{1}{m}(\frac{1}{2}\rho S v^2 u_2\cos\theta + (u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l_0})^2))\sin\theta - W) + \frac{p_1}{m}(u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l0})^2) - W\sin\theta) + \frac{p_2}{m}(\frac{1}{2}\rho S v u_2 - \frac{W}{v}\cos\theta) \quad (19)$$

And the co-state equations are obtained by $\dot p_1 + \frac{\partial H}{\partial v} = 0$ and $\dot p_2 + \frac{\partial H}{\partial \theta} = 0$ as follows:

$$\dot p_1 + \frac{1}{m}(\rho S v u_2\cos\theta - \rho S v(C_{D_0} + K(u_2 - C_{l_0})^2)\sin\theta + p_1(-\rho S v(C_{D_0} + K(u_2 - C_{l_0})^2)) + p_2(\frac{1}{2}\rho S u_2 + \frac{W}{v^2}\cos\theta)) = 0 \quad (20)$$

$$\dot p_2 + \frac{1}{m}(-\frac{1}{2}\rho S v^2 u_2\sin\theta + (u_1 - \frac{1}{2}\rho S v^2(C_{D_0} + K(u_2 - C_{l_0})^2))\cos\theta - p_1 W\cos\theta + p_2(\frac{W}{v}\sin\theta)) = 0 \quad (21)$$

The condition on Hamiltonian for optimality is:

$$H(x^*(t), u^*(t), p^*(t), t) \ge H(x^*(t), u(t), p^*(t), t) \quad (22)$$

Thus separation of all terms involving u_1 yields:

$$\frac{1}{m}(p_1 + \sin\theta)u_1$$

Hence, the necessary and sufficient conditions for unbounded u_1 is:

$$u_1^* = \begin{cases} +\infty & \text{for } (p_1 + \sin\theta) \succ 0 \\ -\infty & \text{for } (p_1 + \sin\theta) \prec 0 \end{cases} \quad (23)$$

Thus, the value of u_1 is not known at this point for the singular condition $p_1 + \sin\theta = 0$. (22) also requires that $\frac{\partial H}{\partial u_2} = 0$ and $\frac{\partial^2 H}{\partial u_2^2} \prec 0$. Therefore:

$$m\frac{\partial H}{\partial u_2} = \frac{1}{2}\rho S v^2\cos\theta - \rho S v^2 K(u_2 - C_{l_0})\sin\theta + p_1(-\rho S v^2 K(u_2 - C_{l_0})) + p_2(\frac{1}{2}\rho S v) = 0 \Rightarrow$$

$$u_2^* = C_{l_0} + \frac{v\cos\theta + p_2}{2vK(\sin\theta + p_1)} \quad (24)$$

Now the systems of differential equations (16), (20), (21) must be solved which if not impossible, is a highly unreliable task to do numerically, even if the boundary values were to be given for that the derivatives of the co-states (and the states) are unbounded at the initial point.

Nevertheless, we claim that the analytical solutions of the system of two-point boundary values are as follows:

$$p_1(t) = -\sin\theta(t) \quad (25)$$
$$p_2(t) = -v(t)\cos\theta(t) \quad (26)$$

Next it will be shown that these equations do satisfy the system of differential equations. For the proposed solution of (25) and (26):

$$\dot p_1 = -\dot\theta\cos\theta \quad (27)$$
$$\dot p_2 = -\dot v\cos\theta + v\dot\theta\sin\theta \quad (28)$$

Substituting from (16):

$$\dot p_1 = -\frac{1}{m}(\frac{1}{2}\rho S v u_2\cos\theta - \frac{W}{v}\cos^2\theta) \quad (29)$$

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$$\dot{p}_2 = \frac{1}{m}\left(\frac{1}{2}\rho S v^2 u_2 \sin\theta - \left(u_1 - \frac{1}{2}\rho S v^2 \left(C_{D_0} + K(u_2 - C_{l_0})^2\right)\right)\cos\theta\right) \quad (30)$$

It is straightforward that (25), (26), (29) and (30) satisfy (20) and (21).

Singularity and Controllability

(25) demonstrates that $p_1 + \sin\theta = 0$. Thus (23) suggests that the system is singular which is not entirely unpredictable for minimum-time problems. Substituting (25) and (26) in (24) yields $\frac{0}{0}$. Hence the limiting behavior must be studied. However it should be further investigated whether the right limit or the left limit yields the minimum-time, i.e. whether $\underset{y_1 \to 1^+}{y_1 p_1}$ should be considered or $\underset{y_1 \to 1^-}{y_1 p_1}$.

In addition to $\frac{\partial H}{\partial u_2} = 0$, (22) also implies that $\frac{\partial^2 H}{\partial u_2^2} \prec 0$. Therefore the following must be true:

$$\frac{\partial^2 H}{\partial u_2^2} = -\rho S v^2 K(\sin\theta + p_1) \prec 0 \quad (31)$$

For (31) to be true, since $0 \le \theta \le \frac{\pi}{2}$, it must be that:

$$p_1 = \underset{y_1 \to 1^-}{-y_1 \sin\theta} \quad (32)$$

So that $\sin\theta + p_1 = \sin\theta(1 - y_1)$ is positive and (31) is satisfied. Thus minimum-time requires the right limit of p_1 and by (23):

$$u_1^* = \begin{cases} +\infty .... for ...(p_1 + \sin\theta) \to 0^+ \\ -\infty ... for ....(p_1 + \sin\theta) \to 0^- \end{cases} \quad (33)$$

Hence:

$$p_1 + \sin\theta = \sin\theta(1 - y_1) \succ 0 \Rightarrow \tilde{u}_1^* = +\infty$$

Therefore, for bounded control $0 \le u_1 \le T_{\max}$, motor thrust is saturated to maximum level during minimum-time maneuver which should have been obvious from the beginning.

The behavior of p_2 should also be examined near the singular point for the right limit of p_1:

$$p_2 = -y_2 v \cos\theta \quad (34)$$

Since p_1 and p_2 are related through (20) and (21), $y_2(y_1)$ is unknown, however considering (25) and (26) the following is true for y_2 :

$$\lim_{y_1 \to 1} y_2(y_1) = 1 \quad (35)$$

Thus the Taylor series expansion around $y_1 = 1$ is:

$$y_2(y_1) = 1 + y_2'(1)(y_1 - 1) + \frac{y_2''(1)(y_1 - 1)^2}{2!} + ... \quad (36)$$

Substituting (25) and (26) in (24) yields:

$$u_2^* = C_{l_0} + \lim_{y_1 \to 1^-} \frac{v\cos\theta(1 - y_2)}{2vK\sin\theta(1 - y_1)} \quad (37)$$

Now, by substitution of (36), simplifying and taking the limit, (37) gives:

$$u_2^* = C_{l_0} + \frac{1}{2K} y_2'(1)\cot\theta = C_{l_0} + \mu\cot\theta \quad (38)$$

where μ is a constant depending only on the specified boundary value $z(t_f) = c$ (Different optimal controls for different boundary values result in different states in (16) upon which depend p_1 and p_2 and hence $y_2'(1)$). So the family of solutions for different boundary values is given by (38). The function $\mu(c)$ will be depicted in simulations. The useful attribute of (38) is in that the influence of the co-states p_1 and p_2 have been reduced to a single constant μ. Furthermore since the unbounded control is found by (38), for a given boundary value, bounded controls are easily found by the following:

$$\tilde{u}_2^* = \begin{cases} c_{\max} & c_0 + \mu\cot\theta \ge c_{\max} \\ c_0 + \mu\cot\theta & c_{\min} \le c_0 + \mu\cot\theta \prec c_{\max} \\ c_{\min} & c_0 + \mu\cot\theta \prec c_{\min} \end{cases} \quad (39)$$

The last condition is obviously never met for $\mu \succ 0$. Now the set of controls $u_v^*(t_{f,a}, t)$ given by (39) span U' as defined earlier for $t_{f,a} \ge 0$. With thrust motor set to $T_{\max}$, $v(u_v^*(t_{f,a}, t), t)$ is always increasing. Then as suggested by theorem 1, the optimal control is determined by (39) and the following:

$$t^* = \inf_{\mu}\{t : z(t,u) = c\} \blacksquare$$

IV. SIMULATIONS

The UAV used for simulations here is a CAP 232 0.90 aerobatic aircraft whose parameters are excerpted from [11] and summarized in Table 1. Other parameters used for example simulations are given in Table 2. Parameters of Table 2 might not be real-world parameters. Yet they have been used here to demonstrate the numerical results of the mathematical concepts.

TABLE 1: AIRCRAFT PARAMETERS

Parameter:	Mass (kg)	$T_{\max}$ (N)	S (m^2)	C_{D_0}	C_{l_0}
Value:	5	70	0.5	0.02	0

TABLE 2: EXAMPLE PARAMETERS FOR SIMULATION

Parameter	initial speed (m/s)	initial angle of trajectory (rad)	air density (kg/m^3)	K in drag polar	$C_{l,\max}$	$C_{l,\min}$
Value:	30	0	1	0.1	1.7	-1.7

The function $\mu(c)$ of (38) for free $\theta(t_f)$ is depicted in Fig. 3. As is evident in the figure, the further the waypoint (bigger c), the less effort the second control (lift coefficient) puts after the maneuver starts. The reason for this is to let the angle of trajectory remain far from $\frac{\pi}{2}$ for a longer time. To see this, the trajectory for three different final boundary values c=30m , c=175m , c=1000m are shown in Fig. 4. Their altitude (z(t)) are depicted in Fig 5. The time it takes for the trajectory, optimal at c=30m to reach the waypoint c=1000m is 20.7s. In other words $t(u_z^*(30),1000) = 20.7s$, while $t^* = 19.03s$ and $t(u_z^*(175),1000) = 19.93s$.

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Optimal time for different values of boundary value c is depicted in Fig. 6.

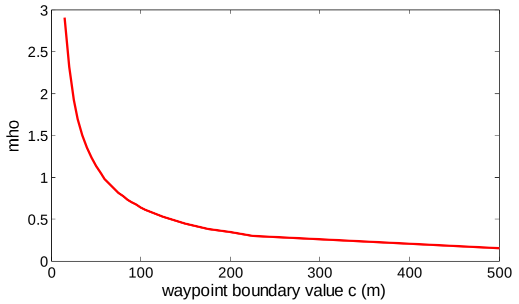

Figure 3. μ(c) for UAV parameters of Table 1 and Table 2

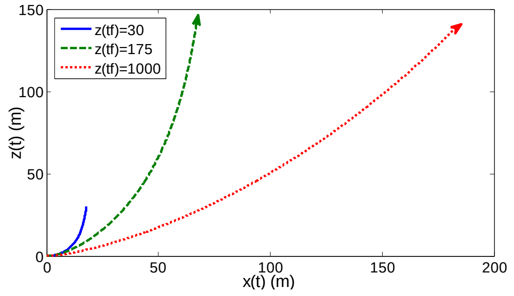

Figure 4. Trajectory for different $z(t_f)$ for free $\theta(t_f)$

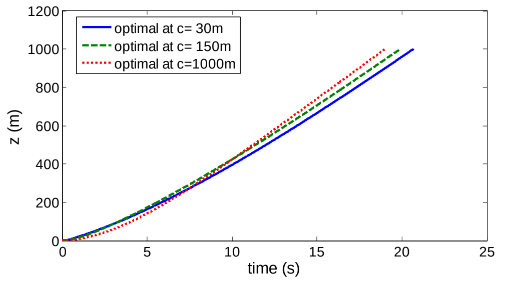

Figure 5. Height (z(t)) for different $t_{f,a}$, optimal at $z(t_f) = 30,\ 175,\ 1000$ m for free $\theta(t_f)$

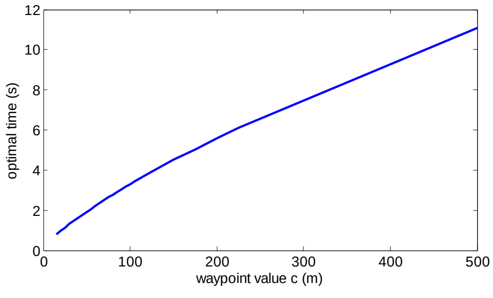

Figure 6. Optimal time versus $z(t_f)$ for free $\theta(t_f)$

In all the figures that follow, different variables are depicted both for free $\theta(t_f)$ and $\theta(t_f)=0$. Also $z(t_f)=c=100$m.

The optimal trajectory for free $\theta(t_f)$ and that of the proposed method for $\theta(t_f)=0$ are shown in Fig. 7. As is illustrated in the figure, for the specified final angle, the trajectory makes more effort to rise and then decreases its angle.

First thing that catches the eye is that the optimal time for free $\theta(t_f)$ is 3.3s while the time for $\theta(t_f)=0$ with proposed method satisfying an extra boundary value is 3.515s. This perhaps suggests that the method may in fact be giving the global optimum. Also the variables mentioned in the proposed method are t_s=3.23s and t_r=3.7s meaning that the control in the first phase of the maneuver optimizes z at t=3.27s regardless of final boundary values.

The respective controls for these maneuvers are depicted in Fig. 8. Lift and drag are depicted in Fig. 9.

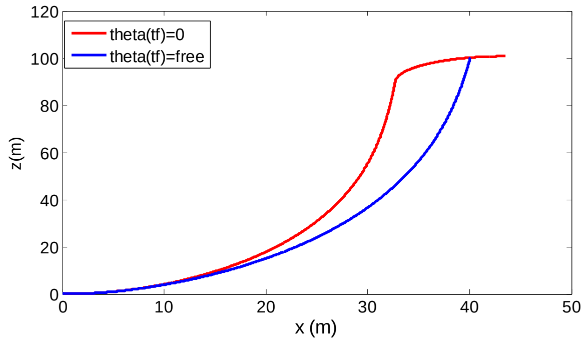

Figure 7. Trajectory of $z(t_f) = 100\,\text{m}$ for $\theta(t_f) = 0$, free

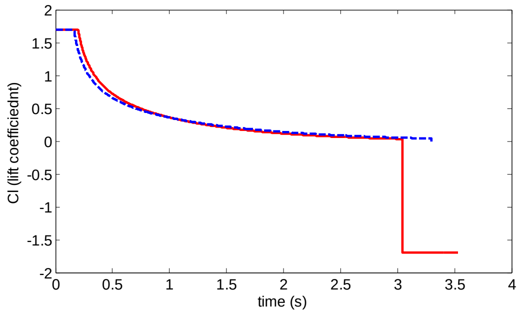

Figure 8. Control of $z(t_f) = 100\,\text{m}$ for $\theta(t_f) = 0$, free

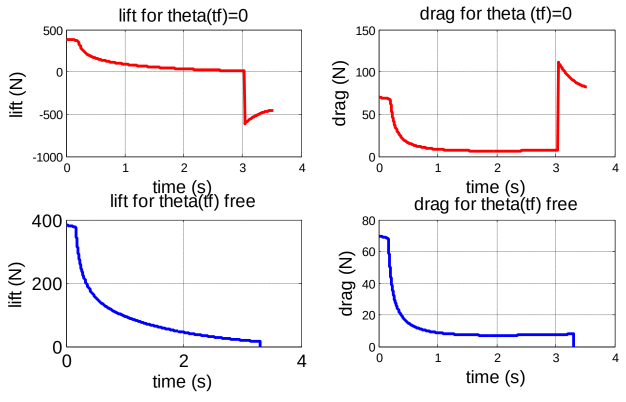

Figure 9. Lift and drag of $z(t_f) = 100\,\text{m}$ for $\theta(t_f) = 0$, free

The inertial z-component of the resultant force and jerk are shown in Fig. 10. It is clear from the figure that jerk is negative which was required by theorem 1.

True speed and the angle of trajectory are illustrated in Fig. 11. One aspect of the methods proposed here is that they work for any initial angle $0 \le \theta(0) \le \frac{\pi}{2}$ and not just $\theta(0) = 0$. Trajectory and control are depicted in Fig. 12 for the initial

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angle angle $\theta(0)=\frac{\pi}{6}$ and $\theta(t_f)=$ free and $\theta(t_f)=\frac{\pi}{3}$. Trajectory angle during maneuver is illustrated in Fig 13.

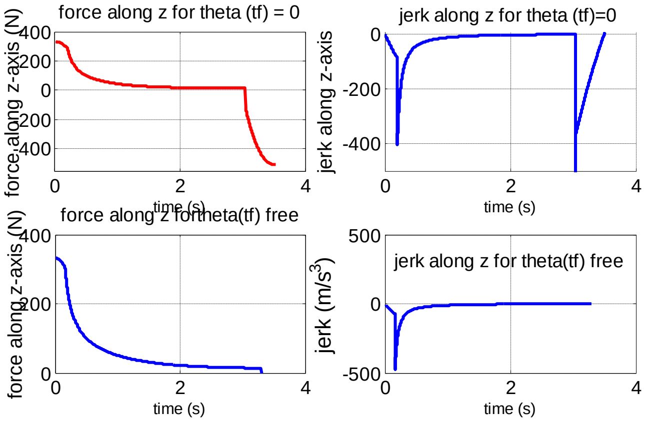

Figure 10. z-axis force and jerk of $z(t_f)=100\,\text{m}$ for $\theta(t_f)=0$, free

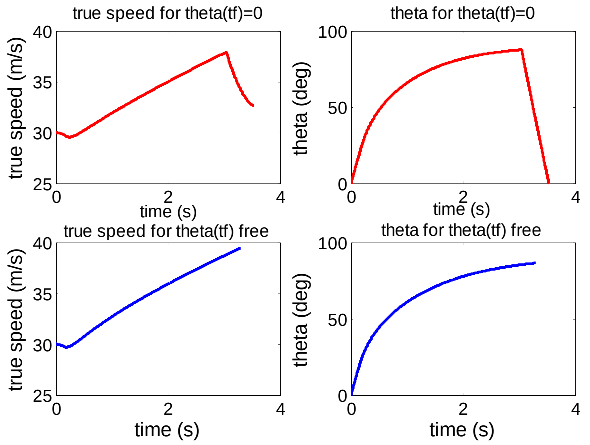

Figure 11. Speed and trajectory angle of $z(t_f)=100\,\text{m}$ for $\theta(t_f)=0$, free

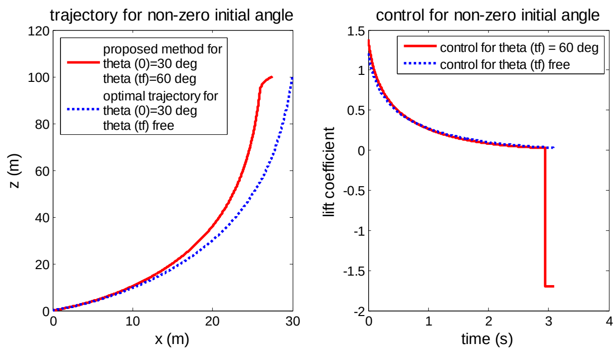

Figure 12. Trajectory of non-zero initial angle of $z(t_f)=100\,\text{m}$

for $\theta(t_f)=\frac{\pi}{3}$, free

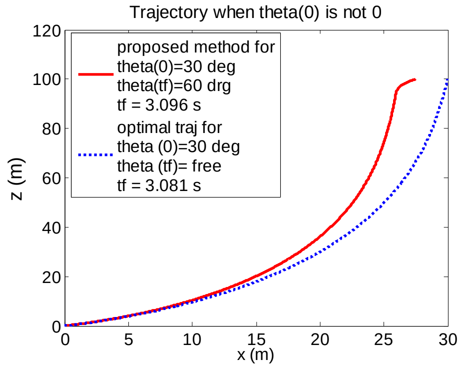

Figure 13. Angle of trajectory of non-zero initial angle of $z(t_f)=100\,\text{m}$

for $\theta(t_f)=\frac{\pi}{3}$, free

For free $\theta(t_f)$, the optimal control was found earlier. However for $\theta(t_f)=0$, some comparisons are made between the proposed method, the local optimum obtained by bang-bang control, and the optimal control for the maneuver starting at the point from where the maneuver requires zero motor thrust at $t=0$ as suggested by [12]. This will be referred to in what follows as method 2.

For method 2, the appropriate final boundary condition is $v(t_f)=0$ so that the aircraft will have the ability to resume maneuver horizontally at the waypoint [12]. It should be noted that the angle of trajectory is not continuous at the waypoint.

The trajectories and z(t) are shown in Fig. 14. The final time for the proposed method in here is 3.515s, for the best bang-bang method 3.75s and for method 2 is 6.052s. The proposed method of this paper shows 6.63% improvement over bang-bang control.

Controls are depicted in Fig. 15. The method of this paper is the smoothest of all especially for free $\theta(t_f)$ for which the proposed method yields a continuous control on $t \succ 0$

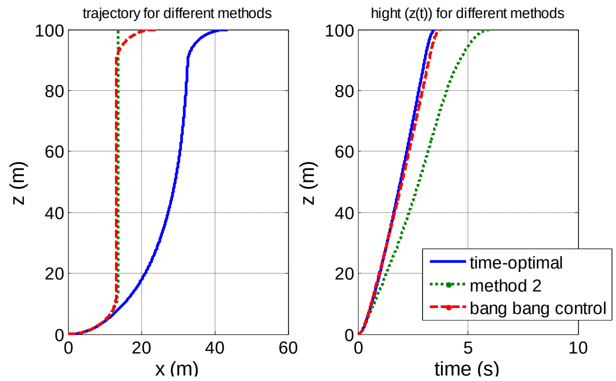

Figure 14. Trajectory for Three Different Controls

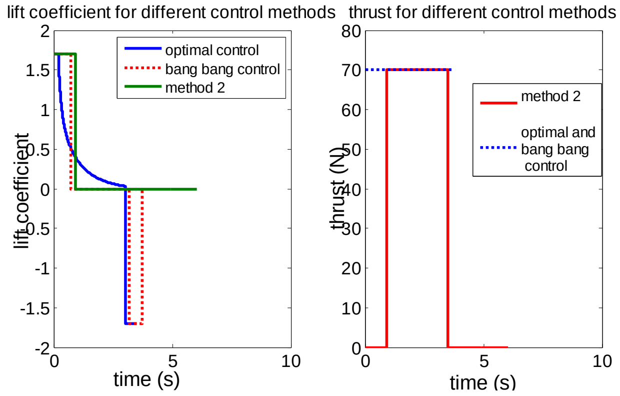

Figure 15. Control for Three Different Methods

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